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ON L^p-ESTIMATES FOR MAXIMAL AVERAGES OVER HYPERSURFACES NOT SATISFYING THE TRANSVERSALITY CONDITION

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Abstract

This thesis mainly deals with L^p-estimates of maximal functions associated with hypersurfaces located at the origin in $\mathbb{R}^3$. The general problem in $\mathbb{R}^n$ is stated as follows. Let $\Gamma \subseteq \mathbb{R}^n$ be a smooth hypersurface and let $\psi \in C_0^\infty(\Gamma)$ be a smooth positive function with compact support. Denote by $d\sigma$ the standard surface measure on Γ. The maximal operator $\mathcal{M}$, initially defined on Schwartz functions, is given by

$$\mathcal{M}f(x) = \sup_{t>0} \left| \int_\Gamma f(x - ty)\psi(y)d\sigma(y) \right|, \quad x \in \mathbb{R}^n.$$

The goal is to determine for which p the maximal operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$. The introduction motivates the study of maximal functions and provides an overview of the previous results.

Hypersurfaces located at the origin do not satisfy the transversality assumption on the hypersurface Γ introduced by I. Ikromov, M. Kempe and D. Müller, saying that for every point $x^0 \in \Gamma$, the affine tangent plane $x^0 + T_{x^0}\Gamma$ does not pass through the origin. We prove that the maximal averages over analytic hypersurfaces located at the origin in general behave more regularly than the maximal averages over hypersurfaces satisfying the transversality condition. In this context we discuss two different conjectures and their confirmations – in particular, the conjecture of A. Iosevich and E. Sawyer which relates the range of p for which the L^p-boundedness of $\mathcal{M}$ holds with the contact order of the hypersurface Γ to its tangent planes not passing through the origin.

We use the well known oscillatory integrals techniques, establishing several crucial uniform estimates for different classes of oscillatory integrals with small parameters, mainly using the method of stationary phase and the van der Corput lemma. Rather than using damping functions like the Gaussian curvature we develop an algorithm adapted

to the problem for the resolution of singularities of an analytic function in two variables. This algorithm does not cover a particular degenerate situation, where we shall adapt the ideas of the stopping time procedure developed by I. Ikromov, M. Kempe and D. Müller.

It turns out that the established L^p-estimate of the maximal operator is in general not true if the hypersurface is not assumed to be analytic but only to be smooth and of finite type. We point out the importance of the finite type condition, giving an example of a flat curve at the origin such that the corresponding maximal average is only bounded on $L^\infty(\mathbb{R}^2)$.

Zusammenfassung

In dieser Arbeit werden L^p-Abschätzungen für Maximalfunktionen über glatte Hyperflächen im $\mathbb{R}^3$ behandelt. Das allgemeine Problem lässt sich folgendermaßen formulieren: Sei $\Gamma \subseteq \mathbb{R}^n$ eine glatte Hyperfläche und sei $\psi \in C_0^\infty(\Gamma)$ eine glatte positive Funktion mit kompakten Träger. Sei $d\sigma$ das Oberflächenmaß auf Γ. Der Maximaloperator $\mathcal{M}$ ist definiert durch

$$\mathcal{M}f(x) = \sup_{t>0} \left| \int_\Gamma f(x - ty)\psi(y)d\sigma(y) \right|, \quad x \in \mathbb{R}^n,$$

wobei f zunächst eine Schwartzfunktion sei. Obwohl es viele Arbeiten auf diesem Gebiet gibt, ist bislang kein vollständiges Ergebnis für die L^p-Beschränktheit von $\mathcal{M}$ bekannt. In der Einleitung ist u. a. der aktuelle Stand der Forschung beschrieben.

Analytische Hyperflächen im Ursprung genügen nicht der Transversalitätsbedingung, die von I. Ikromov, M. Kempe und D. Müller eingeführt wurde. Sie besagt, dass für jeden Punkt x^0 auf der Hyperfläche Γ die affine Tangentialebene $x^0 + T_{x^0}\Gamma$ nicht durch den Ursprung verläuft. In dieser Arbeit wird gezeigt, dass die Maximalfunktionen über analytischen Hyperflächen im Ursprung im Allgemeinen einer besseren Abschätzung auf $L^p(\mathbb{R}^3)$ genügen als die Maximalfunktionen über die Hyperflächen, welche die Transversalitätsbedingung erfüllen. In diesem Kontext werden zwei wichtige Vermutungen diskutiert, darunter die Vermutung von A. Iosevich und E. Sawyer, welche das Interval der Elemente p, für die der Maximaloperator $\mathcal{M}$ auf L^p-beschränkt ist, in Verbindung bringt mit der Kontaktordnung der Hyperfläche zu ihren affinen Tangentialebenen, die nicht durch den Ursprung verlaufen.

In dieser Arbeit wird die bekannte Methode der oszillierenden Integrale verwendet.
Hierfür werden für einige Klassen oszillierender Integrale mit Phasenfunktion, die noch von zusätzlichen kleinen Parametern abhängt, uniforme radiale Abschätzungen bewiesen.

Hierbei werden hauptsächlich die bekannten Methoden der stationären Phase und des van der Corput Lemmas verwendet. Statt mit Dämpfungsfaktoren zu arbeiten, wird im Beweis ein an das Problem adaptiertes Verfahren entwickelt, um die Singularitäten einer analytischen Funktion in zwei Variablen präzise aufzulösen. Dieses Verfahren deckt eine spezielle, ausgeartete Situation, die sich mit Hilfe der Geometrie der Newton-Diagramme beschreiben lässt, nicht ab. Hierfür wird ein anderes Verfahren, welches ursprünglich von I. Ikromov, M. Kempe und D. Müller entwickelt wurde, an diese degenerierte Situation angepasst.

Tatsächlich ist im Allgemeinen das Verhalten der Maximalfunktionen über glatte Hyperflächen von endlichem Typ erheblich schlechter als für die analytischen Hyperflächen. Außerdem zeigt eine Konstruktion einer speziellen glatten Kurve, die nicht von endlichem Typ am Ursprung ist, dass die zugehörige Maximalfunktion nur auf $L^\infty(\mathbb{R}^2)$ beschränkt ist.

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Contents

1 Introduction

1.1 Introduction to maximal functions and their role in questions of pointwise convergence of integral averages

One general question concerning the limit behavior of integral averages can be formulated as follows. Given a set $\Sigma \subseteq \mathcal{P}(\mathbb{R}^n)$ of measurable sets S and a measure μ, how regular does a measurable complex-valued function f have to be so that

$$\lim_{\mu(S)\to 0} \frac{1}{\mu(S)} \int\limits_S f(x-y)d\mu(y) = f(x)$$

holds true for almost all $x \in \mathbb{R}^n$? In the theory of integral operators the regularity properties of a function are usually measured in terms of an integrability condition or more generally in terms of Sobolev spaces.
A simple example arises by considering the integral averages over symmetric intervals in $\mathbb{R}$ containing the origin, namely, let the average $Af(x,r)$ of a measurable function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ over the interval $[-r,r]$ at the point $x \in \mathbb{R}$ be defined by

$$Af(x,r) = \frac{1}{2r} \int\limits_{-r}^{r} f(x-y)dy.$$

It is well known (Lebesgue's differentiation theorem) that for any locally integrable function f on $\mathbb{R}$

$$\lim_{r\to 0} \frac{1}{2r} \int\limits_{-r}^{r} f(x-y)dy = f(x)$$

holds true for almost all $x \in \mathbb{R}$. More general questions can be stated in any dimension also for rectangles $R \subseteq \mathbb{R}^n$ centered at the origin. Then the integral average $A_R f(x)$ over such a rectangle R is given by

$$A_R f(x) = \frac{1}{|R|} \int\limits_R f(x-y)dy.$$

It is a very interesting phenomenon that already in $\mathbb{R}^2$ the pointwise convergence may fail even for characteristic functions if the limit is taken over the family of all rectangles with arbitrary directions. On the other hand, if the family of all rectangles is replaced by a more restricted family of sets, namely rectangles with sides parallel to the coordinate axes, the limit behavior is well known to be regular on $L^p(\mathbb{R}^2)$ for any $p > 1$.
The proof for the negative result is complicated and is connected to Besicovitch sets. A Besicovitch set is a subset of $\mathbb{R}^2$ (or more generally of $\mathbb{R}^n$) which contains a unit line segment in each direction. A. Besicovitch constructed such a set of measure zero.
These questions already indicate that the geometry and the dimension of the underlying sets play a significant role for the questions of pointwise convergence. The important tool for the study of pointwise limit behavior is the corresponding maximal average

$$\mathcal{M}_\Sigma f(x) = \sup_{S \in \Sigma} \frac{1}{\mu(S)} \int\limits_S |f(x-y)| \, d\mu(y).$$

The well known approach can be described in the following general setup.
Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two measure spaces and let p, $q \in [1, \infty)$. Let D be a dense subspace of $L^p(X)$ and suppose that for every $\varepsilon > 0$

$$T_\varepsilon \colon L^p(X) \longrightarrow L^0(Y)$$

is a linear operator defined on $L^p(X)$ with values in the set of all measurable functions $L^0(Y)$ on Y. Define the maximal operator T^* by

$$T^* f(y) = \sup_{\varepsilon > 0} |T_\varepsilon f(y)|, \quad f \in L^p(X), \; y \in Y.$$

Obviously T^* is a sublinear operator. Assume that T^* is of weak type (p,q), i.e. there exists a constant $C_{p,q} \geq 0$ such that

$$\sup_{\lambda>0} \lambda \left(\nu \{y \in Y : \ |T^* f(y)| > \lambda\}\right)^{\frac{1}{q}} \leq C_{p,q} \|f\|_{L^p(X)}$$

holds true for any function $f \in L^p(X)$. If $\lim\limits_{\varepsilon \to 0} T_\varepsilon(f)$ exists ν-a.e. and is finite for any $f \in D$, then the same holds true for every function $f \in L^p(X)$. In other words, the linear operator

$$Tf(y) = \lim_{\varepsilon \to 0} T_\varepsilon f(y),$$

initially defined on D, can be uniquely extended to a bounded operator from $L^p(X)$ to $L^{q,\infty}(Y)$. For the proof and further details we refer the reader to [11].
This shows that the questions of pointwise convergence of the above integral averages are strictly connected to the L^p-estimates for the corresponding maximal functions. In fact, in a certain sense the converse also holds true for a large class of integral averages. The existence of the limit already implies that the maximal average is of weak-type (p,p). For more details we refer the reader to [43].

1.2 History of the problem

A wide-ranging generalization of the examples from the previous section is obtained by considering averages over lower-dimensional sets in $\mathbb{R}^n$, in particular over smooth hypersurfaces. Let Γ be a smooth hypersurface in $\mathbb{R}^n$. Let $\psi \in C_0^\infty(\Gamma)$ be a smooth positive function with compact support. Denote by $d\sigma$ the standard surface measure on Γ and set $d\mu = \psi d\sigma$. For $t \in \mathbb{R}$ consider the usual isotropic dilation given by

$$\mu_t(f) = \int\limits_\Gamma f(ty)\mu(y), \quad f \in \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz space. Let A be the associated averaging operator given by

$$Af(x,t) = \int\limits_\Gamma f(x-ty)\mu(y) = f * \mu_t(x), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

The associated maximal operator $\mathcal{M}$, initially defined on the Schwartz space $\mathcal{S}$, is given by

$$\mathcal{M}f(x) = \sup_{t>0} |Af(x,t)| .$$

It is an obvious fact that $\mathcal{M}$ is a sublinear operator bounded on $L^\infty(\mathbb{R}^n)$. Furthermore, since

$$\lim_{t\to 0} Af(x,t) = \mu(\Gamma)f(x)$$

holds true for any Schwartz function f and $x \in \mathbb{R}^n$, we conclude that $\mathcal{M}$ is unbounded as an operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $q \neq p$. The goal is to determine the "minimal" $p_0 > 1$ (the range of p for which $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ can be open) such that $\mathcal{M}$ is bounded on $L^{p_0}(\mathbb{R}^n)$.

The earliest work in this area was done for the case of the Euclidean unit sphere in $\mathbb{R}^n$, $n \geq 3$, when in 1976 E. M. Stein [42] proved that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$. A necessary condition for the L^p-boundedness of the maximal average over the unit sphere is easily seen to be $p > \frac{n}{n-1}$ in any dimension by testing $\mathcal{M}$ on the characteristic function of the unit ball in $\mathbb{R}^n$. A. Greenleaf proved that the maximal operator $\mathcal{M}$ is also bounded on $L^p(\mathbb{R}^n)$ for $n \geq 3$ and $p > \frac{n}{n-1}$ if the hypersurface has everywhere non-vanishing Gaussian curvature and satisfies an additional geometric condition [18]. The case $n = 2$ was later proved by J. Bourgain [4]. In these articles the role of the curvature and the geometry of the surface became evident. On the other hand, if at each point of the hypersurface Γ the Gaussian curvature does not vanish to infinite order, C. D. Sogge and E. M. Stein used damping techniques to show that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$, $n \geq 3$, in a certain range $p > p_0$ [41]. In general, this exponent p_0 is far from being optimal. C. D. Sogge proved in [40] that the maximal operator is bounded on $L^p(\mathbb{R}^n)$ for $p > 2$ whenever the hypersurface has at least one non-vanishing principal curvature everywhere. Sharp results in higher dimensions for hypersurfaces with vanishing Gaussian curvature are only known for particular classes of hypersurfaces.

The case of finite-type curves in $\mathbb{R}^2$ was studied by A. Iosevich [23]. Convex hypersurfaces in higher dimensions have also been studied extensively. We refer the reader to the work of A. Nagel, A. Seeger and S. Wainger [35], the articles [25], [26] of A. Iosevich and E. Sawyer and A. Iosevich, E. Sawyer and A. Seeger [27].

As observed by E. M. Stein, the uniform decay rate of the Fourier transform of the measure

$$\widehat{\mu}(\lambda) = \int_{\Gamma} e^{-i\lambda\cdot y}\mu(y), \quad \lambda \in \mathbb{R}^n,$$

plays a fundamental role for the estimates of $\mathcal{M}$, i.e. the problem of L^p-regularity of the maximal averages is connected to the oscillatory integral estimates, see e.g. Theorem 4.0.13. If on $\operatorname{supp}\psi$ the hypersurface Γ is locally parametrized by $\Phi(u) = (u, \phi(u))$ and the function ϕ is not flat, then it is well known by the van der Corput lemma that there must be some uniform estimate for the decay of $\widehat{\mu}$, when $|\lambda| \to \infty$. However, in general, the asymptotic behavior of oscillatory integrals is very difficult to understand.
I. Ikromov, M. Kempe and D. Müller discovered in [20], [21] a connection between the height of a surface in $\mathbb{R}^3$ and the L^p-boundedness of the corresponding maximal average. The study was done under a transversality assumption on the underlying hypersurface Γ, saying that for every point $x_0 \in \Gamma$, the affine tangent plane $x_0 + T_{x_0}\Gamma$ does not pass through the origin. In particular, $0 \notin \Gamma$. In such a case, the L^p-boundedness for $p > 2$ of the local maximal average at some point $x_0 \in \Gamma$ is determined by the height $h(x_0, \Gamma)$ of Γ at the point x_0. For the precise definition of $h(x_0, \Gamma)$ we refer the reader to [21].

1.3 Statement of main results

In this thesis we shall investigate the regularity of the maximal average over a hypersurface at the origin, so in particular it does not satisfy the transversality assumption. The main result is the next theorem.

Theorem 1.3.1. *Suppose Γ is a hypersurface in $\mathbb{R}^3$ which is parametrized as the graph of a real-valued analytic function $\varphi\colon U \longrightarrow \mathbb{R}$ at the origin, i.e.*

$$\Gamma = \{(x, \varphi(x)) : \ x = (x_1, x_2) \in U\}. \tag{1.1}$$

Let $\Phi(x) = (x, \varphi(x))$. We also assume that

$$\varphi(0,0) = 0, \quad \nabla\varphi(0,0) = (0,0).$$

Then there exists a neighborhood of the origin $\Omega \subseteq U$ such that for every positive smooth *function* $\psi \in C_0^\infty(\Omega)$ *the associated maximal operator*

$$\mathcal{M}f(z) = \sup_{t>0} \left| \int_\Omega f(z - t\Phi(x))\psi(x)dx \right|, \quad z \in \mathbb{R}^3, \tag{1.2}$$

initially defined on the Schwartz space $\mathcal{S}(\mathbb{R}^3)$*, is bounded on* $L^p(\mathbb{R}^3)$ *for every* $p > 2$*.*

In fact, the proof of the above theorem will also cover a large class of maximal operators, where the standard dilation is replaced by a more general family of the non-isotropic dilations D_t^a given by

$$D_t^a(x) = (t^{a_1}x_1, t^{a_2}x_2, t^{a_3}x_3), \quad t > 0, \tag{1.3}$$

where a_1, a_2, a_3 are positive real numbers. For this class of maximal operators we also obtain the boundedness on $L^2(\mathbb{R}^3)$.
It is remarkable that the above L^p-estimate for the maximal operator (1.2) is in general not true if the hypersurface is not assumed to be analytic but only to be smooth and of finite type. In Chapter 6 we shall prove that for every $m \in \mathbb{N}$ there exists a smooth function ϕ_m which is of finite type along every line at the origin (cf. Definition 6.2.1) such that the associated maximal average over the hypersurface given by the graph of ϕ_m is unbounded on $L^m(\mathbb{R}^3)$. An example of a flat surface at the origin is also discussed in Chapter 6. In Section 6.3 we shall discuss the sharpness of Theorem 1.3.1. The case $p \leq 2$ is work in progress.
We mention that under additional strong assumptions on the hypersurface M. Greenblatt [17] could show by using damping functions that the maximal operator is also bounded on $L^p(\mathbb{R}^3)$ for $p > \max\{2, h(x_0, \Gamma)\}$ not using the transversality assumption.
In this thesis we will not use damping function techniques. The proof is more in the spirit of the article [21] and will be based on a very fine resolution of singularities, involving the geometry of Newton diagrams.
Theorem 1.3.1 gives confirmation to two different conjectures, at least for the analytic class of hypersurfaces at the origin in $\mathbb{R}^3$. The first conjecture is that if the oscillatory integral $\widehat{\mu}$ satisfies

$$\widehat{\mu}(\lambda) = \mathcal{O}(|\lambda|^{-\alpha}), \quad \text{as} \quad |\lambda| \longrightarrow \infty,$$

and $\alpha \leq \frac{1}{2}$, then the maximal operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$, whenever $p > \alpha^{-1}$. The case $\alpha = \frac{1}{2}$ was conjectured by E. M. Stein. Later A. Iosevich and E. Sawyer [26] extended the conjecture to the range $\alpha \leq \frac{1}{2}$. Thus Theorem 1.3.1 obviously confirms this conjecture for the analytic class of hypersurfaces at the origin in $\mathbb{R}^3$, since $\alpha^{-1} \geq 2$. We remark that the Stein-Iosevich-Sawyer conjecture was also confirmed by I. Ikromov, M. Kempe and D. Müller in [21] for the smooth compact hypersurfaces in $\mathbb{R}^3$ satisfying the transversality assumption.

Another conjecture of A. Iosevich and E. Sawyer relates the range of p for which the L^p-boundedness of $\mathcal{M}$ holds with the contact order of the hypersurface Γ to its tangent planes. More precisely, in [25] it was shown that a necessary condition for the L^p-boundedness of $\mathcal{M}$ is that the function

$$\Gamma \ni x \longmapsto \frac{1}{\operatorname{dist}(x, H)^{\frac{1}{p}}} \tag{1.4}$$

is locally integrable on Γ for every affine tangent hyperplane H not passing through the origin. In [26] it was conjectured that for $p > 2$ the integrability condition of the function (1.4) is also sufficient for the L^p-boundedness of $\mathcal{M}$. Thus Theorem 1.3.1 implies that for sufficiently small analytic surfaces in $\mathbb{R}^3$ at the origin, the conjecture is trivially true. On the other hand, we can also verify the local integrability condition of the function (1.4) for any $p > 2$. In fact, in view of the assumption that φ is analytic, $\varphi(0,0) = 0$ and $\nabla\varphi(0,0) = (0,0)$, one can show using the Puiseux series expansions of analytic functions that

$$\operatorname{Hess}\varphi(x^0) = 0_{\mathbb{R}^{2\times 2}} \Longrightarrow \nabla\varphi(x^0) = (0,0) \Longrightarrow \varphi(x^0) = 0 \tag{1.5}$$

holds true for any $x^0 \in \Omega$ if Ω is chosen sufficiently small, cf. Lemma A.0.3. Therefore if $\operatorname{Hess}\varphi(x^0) = 0_{\mathbb{R}^{2\times 2}}$, then the affine tangent plane at $(x^0, \varphi(x^0))$ obviously passes through the origin. On the other hand, if $\operatorname{Hess}\varphi(x^0) \neq 0_{\mathbb{R}^{2\times 2}}$, then (1.4) is also locally integrable for any $p > 2$ on some small neighborhood of $(x^0, \varphi(x^0))$. In fact, after a change of coordinates we may assume that the distance between the hypersurface near $(x^0, \phi(x^0))$ and its affine tangent plane is locally given by $p(y_1, y_2) + R(y_1, y_2)$, (y_1, y_2) near the origin, where p is a non-trivial homogeneous polynomial of degree two, and $|R(y_1, y_2)| = \mathcal{O}(\|y\|^3)$. On a sufficiently small neighborhood of the origin the inverse of the function $p + R$ is integrable to the power $\frac{1}{p}$ for every $p > 2$. Therefore Theorem 1.3.1 would be a consequence of Iosevich-Sawyer conjecture.

Except for some specific examples, it is not clear until now, if Theorem 1.3.1 also holds true in dimension $n \geq 4$. Some examples, and also the conjecture of A. Iosevich and E. Sawyer, indicate that at least for mixed homogeneous polynomial surfaces it might be true.

Conjecture 1.3.2. *Let $n \geq 4$. Let $\kappa = (\kappa_1, \ldots, \kappa_{n-1}) \in \mathbb{R}_+^{n-1}$ and let $P \colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ be a κ-homogeneous polynomial of degree one, i.e.*

$$P(r^{\kappa_1} x_1, \ldots, r^{\kappa_{n-1}} x_{n-1}) = rP(x),$$

for every $r > 0$, $x \in \mathbb{R}^{n-1}$. In particular, $P(0) = 0$. Let

$$\Gamma = \left\{(x, P(x)) : x \in (-1,1)^{n-1}\right\}.$$

Then the maximal operator

$$\mathcal{M}f(z) = \sup_{t>0} \left| \int\limits_{\Gamma} f(z - ty) d\sigma(y) \right|$$

is bounded on $L^p(\mathbb{R}^n)$ for every $p > 2$.

In view of Euler's homogeneity relation, see Lemma 2.2.4, and by applying the result of C. D. Sogge in article [40], it is easily seen that the above conjecture is true if ∇P does not vanish away from the origin. The very heart of the problem is therefore to understand the behavior of $\mathcal{M}$ near the zero set of a polynomial in higher dimensions. This research is work in progress.

In this thesis we shall use the well known notation $A \ll B$ if A is much less than B, i.e. if there is a sufficiently large constant K, which is independent of certain underlying relevant quantities, such that $KA \leq B$. We write also $B \gg A$ if $A \ll B$. We write $A \sim B$, and say that A and B are comparable, if neither $B \gg A$ nor $A \gg B$. We write $A \lesssim B$ if $A \ll B$ or $A \sim B$.

2 Preliminaries

In this chapter we shall state and prove some auxiliary results which will be needed in the further proofs. We first recall some basic notions, which essentially go back to the article of A. N. Varchenko [44]. For further details we refer the reader to the articles [21], [22] of I. Ikromov and D. Müller.

2.1 Introduction to Newton diagrams

For a smooth function $\varphi\colon V \longrightarrow \mathbb{R}$ defined on some neighborhood $V \subseteq \mathbb{R}^2$ of the origin we associate its Taylor series centered at the origin

$$\varphi \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} x_1^j x_2^k,$$

where

$$a_{jk} = \frac{\partial_1^j \partial_2^k \varphi(0,0)}{j!k!}.$$

We shall also assume

$$\varphi(0,0) = 0, \quad \nabla\varphi(0,0) = (0,0).$$

If the function φ is analytic at the origin, then clearly

$$\varphi(x_1, x_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} x_1^j x_2^k,$$

and the series converges in a small neighborhood of the origin. Denote by

$$\mathcal{T}(\varphi) = \left\{(j,k) \in \mathbb{N}_0^2 : \ a_{jk} \neq 0\right\}$$

the Taylor support of φ at the origin. Observe that $\mathcal{T}(\varphi) \neq \emptyset$ if φ is analytic and not identically zero. In this section we assume φ to be of finite type at the origin, i.e. $\mathcal{T}(\varphi) \neq \emptyset$. The Newton polyhedron of φ is defined by

$$\mathcal{N}(\varphi) = \operatorname{conv} \bigcup_{(j,k)\in\mathcal{T}(\varphi)} \left((j,k) + \mathbb{R}^2_+\right).$$

The associated Newton diagram $\mathcal{N}_d(\varphi)$ in the sense of Varchenko [44] is the union of all compact faces of $\mathcal{N}(\varphi)$. By a face, we shall mean an edge or a vertex. The Newton distance $d = d(\varphi)$ is determined by the point (d,d), where the bisectrix $\{(t_1,t_2) \in \mathbb{R}^2 : t_1 = t_2\}$ intersects the boundary of the Newton polyhedron, more precisely,

$$d = \inf\{t > 0 : \; (t,t) \in \mathcal{N}(\varphi)\}.$$

The Newton distance can change if we change coordinates. This leads to the notion of the height of the function φ, denoted by $h(\varphi)$. The height of the analytic (smooth) function φ is defined by

$$h(\varphi) = \sup_y d_y,$$

where the supremum is taken over all local analytic (smooth) coordinate systems y at the origin $(0,0)$, and d_y denotes the distance of φ in the coordinates y. A coordinate system y is said to be adapted to φ if $h(\varphi) = d_y$. In [22] was proved that adapted coordinates always exist for smooth functions of finite type, thus generalizing the work by Varchenko [44] who worked in the setting of real-analytic functions.

Example 2.1.1. Consider $\varphi(x_1,x_2) = (x_2 - x_1^2)^2$. Then $d(\varphi) = \frac{4}{3}$. Adapted coordinates are given by

$$(y_1,y_2) = (x_1, x_2 - x_1^2),$$

in which φ is expressed by $\varphi^a(y_1,y_2) = y_2^2$. This gives $h(\varphi) = 2 > \frac{4}{3}$.

In this thesis we will neither need the height nor the distance of an analytic function.

2.2 Auxiliary statements

Lemma 2.2.1. *Let $T\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow L^0(\mathbb{R}^n)$ be a sublinear operator and $1 \leq p < \infty$. Let $A \in \mathrm{GL}(n, \mathbb{R})$. For $f \in \mathcal{S}$ and $x \in \mathbb{R}^n$ set*

$$Sf(x) = T(f \circ A)(A^{-1}x).$$

Then

$$\|T\|_{L^p \to L^p} = \|S\|_{L^p \to L^p}$$

holds true. In particular, T is bounded on $L^p(\mathbb{R}^n)$ if and only if S is bounded on $L^p(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|Sf\|_{L^p}^p &= \int_{\mathbb{R}^n} |T(f \circ A)(A^{-1}x)|^p dx \\
&= \tfrac{1}{|\det A^{-1}|} \int_{\mathbb{R}^n} |T(f \circ A)(x)|^p dx \\
&= |\det A| \cdot \|T(f \circ A)\|_{L^p}^p \\
&\leq |\det A| \cdot \|T\|_{L^p \to L^p}^p \cdot \|f \circ A\|_{L^p}^p \\
&= \|T\|_{L^p \to L^p}^p \cdot \|f\|_{L^p}^p.
\end{aligned}$$

Thus $\|S\|_{L^p \to L^p} \leq \|T\|_{L^p \to L^p}$. Since $S(f \circ A^{-1})(Ax) = Tf(x)$, we conclude that $\|S\|_{L^p \to L^p} \geq \|T\|_{L^p \to L^p}$ using the same arguments. □

Definition 2.2.2. Let $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n$ and let $U \subseteq \mathbb{R}^n$. A function $f\colon U \longrightarrow \mathbb{C}$ is said to be κ-homogeneous of degree $d \in \mathbb{R}$ if

$$f(r^{\kappa_1}x_1, \dots, r^{\kappa_n}x_n) = r^d f(x)$$

for every $x \in U$, $r > 0$ with $(r^{\kappa_1}x_1, \dots, r^{\kappa_n}x_n) \in U$.

Example 2.2.3. The polynomial $P_1(x_1, x_2) = x_1^k + x_2^l$ is $\left(\frac{1}{k}, \frac{1}{l}\right)$-homogeneous of degree one. The polynomial $P_2(x_1, x_2) = x_1^k x_2^l$ is (κ_1, κ_2)-homogeneous of degree $\kappa_1 k + \kappa_2 l$ for

any $(\kappa_1, \kappa_2) \in \mathbb{R}^2$.
Let $\mathcal{H} = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > 0\}$ be the right half-plane, and let $a > 0$. For any function f defined on $\mathbb{R}$ is the function

$$\mathcal{H} \ni (x_1, x_2) \longmapsto f\left(\frac{x_2 - x_1^a}{x_1^a}\right)$$

$\left(\frac{1}{a}, 1\right)$-homogeneous of degree zero.

The next lemma is a version of the Euler's homogeneity relation. The proof can also be found in [20].

Lemma 2.2.4. *Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a κ-homogeneous twice differentiable function of degree one and let $x \in \mathbb{R}^n$. Then*

$$\nabla f(x) = 0 \Longrightarrow f(x) = 0.$$

We also get

$$\operatorname{Hess} f(x) = 0 \Longrightarrow \nabla f(x) = 0$$

if in addition $\kappa_i \neq 1$ for every $i \in \{1, \ldots, n\}$.

Proof. Because of the homogeneity the identity

$$f(r^{\kappa_1} x_1, \ldots, r^{\kappa_n} x_n) = r f(x)$$

holds true for any $r > 0$. Differentiating in r we obtain by the chain rule

$$\nabla f(r^{\kappa_1} x_1, \ldots, r^{\kappa_n} x_n) \cdot (\kappa_1 r^{\kappa_1 - 1} x_1, \ldots, \kappa_n r^{\kappa_n - 1} x_n) = f(x).$$

For $r = 1$ we get

$$\nabla f(x_1, \ldots, x_n) \cdot (\kappa_1 x_1, \ldots, \kappa_n x_n) = f(x).$$

In order to prove the second claim, observe that since f is κ-homogeneous of degree one, the derivative $\partial_i f$ is κ-homogeneous of degree $1 - \kappa_i$. Thus we get

$$\partial_i f(r^{\kappa_1} x_1, \ldots, r^{\kappa_n} x_n) = r^{1-\kappa_i} \partial_i f(x).$$

With the same argument as before we conclude

$$\nabla\partial_i f(x_1,\dots,x_n)\cdot(\kappa_1 x_1,\dots,\kappa_n x_n) = (1-\kappa_i)\partial_i f(x).$$

By assumption $1-\kappa_i \neq 0$ and therefore

$$\frac{\nabla\partial_i f(x_1,\dots,x_n)\cdot(\kappa_1 x_1,\dots,\kappa_n x_n)}{1-\kappa_i} = \partial_i f(x).$$

The desired result follows from the observation that $\nabla\partial_i f$ is the ith row of the Hessian matrix of f. □

The proof of the L^p-boundedness of the maximal operator will be based on an algorithm which consists of certain changes of variables. We have to understand how these transformations effect the geometry of the Newton diagram.
The next lemma describes a connection between the singularities of a κ-homogeneous polynomial in two variables and the weight κ.
If $l_1,\dots,l_n \in \mathbb{N}$, then we use the notation $\gcd(l_1,\dots,l_n)$ for their greatest common divisor.

Lemma 2.2.5. *Let $P\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a κ-homogeneous polynomial of degree one. Assume that P is not of the form $P(x_1,x_2) = c x_1^A x_2^B$, $c\in\mathbb{R}$, $A,\ B\in\mathbb{N}_0$.*
Then the weight κ is uniquely determined by the polynomial P and we have $\kappa_1,\ \kappa_2 \in \mathbb{Q}$. Furthermore,

$$\kappa = (\kappa_1,\kappa_2) = \left(\frac{q}{m},\frac{p}{m}\right), \quad \gcd(p,q)=1.$$

The polynomial P can be factorized in

$$P(x_1,x_2) = C x_1^A x_2^B \prod_{j=1}^{M} (x_2^q - \lambda_j x_1^p)^{n_j},$$

with $M\in\mathbb{N}$, $A,\ B\in\mathbb{N}_0$, $\lambda_j \in \mathbb{C}\setminus\{0\}$ with multiplicities $n_j \geq 1$, $j\in\{1,\dots,M\}$, $C\in\mathbb{R}\setminus\{0\}$.

For the proof we refer the reader to [22].

2.3 On the geometry of Newton polyhedra after a change of coordinates

Let φ be an analytic function in two variables near the origin with

$$\varphi(0,0) = 0, \quad \nabla\varphi(0,0) = (0,0).$$

We assume that the Newton diagram $\mathcal{N}_d(\varphi)$ contains at least one edge. Fix one of those edges and denote it by $\mathfrak{E}$. Consider the Taylor series

$$\varphi(x_1, x_2) = \sum_{\alpha,\beta=0}^{\infty} c_{\alpha,\beta} x_1^\alpha x_2^\beta$$

of φ centered at the origin. The edge $\mathfrak{E}$ lies on the line

$$L_{\tilde{\kappa}} = \{(t_1, t_2) : \ \tilde{\kappa}_1 t_1 + \tilde{\kappa}_2 t_2 = 1\}$$

with the weight $\tilde{\kappa} = (\tilde{\kappa}_1, \tilde{\kappa}_2) \in \mathbb{R}_+^2$ uniquely determined. We can assume that

$$(\tilde{\kappa}_1, \tilde{\kappa}_2) = \left(\frac{q}{m}, \frac{p}{m}\right), \quad \gcd(p,q) = 1.$$

The absolute value of the slope of the line $L_{\tilde{\kappa}}$ is equal to $\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2} = \frac{q}{p}$. Denote by

$$\varphi_{\tilde{\kappa}}(x_1, x_2) = \sum_{(\alpha,\beta)\in\mathfrak{E}\cap\mathbb{N}_0^2} c_{\alpha,\beta} x_1^\alpha x_2^\beta$$

the polynomial corresponding to the edge $\mathfrak{E}$. For $r > 0$ define the dilation $\delta_r^{\tilde{\kappa}}$ associated to the weight $\tilde{\kappa}$ by

$$\delta_r^{\tilde{\kappa}}(x_1, x_2) = (r^{\tilde{\kappa}_1} x_1, r^{\tilde{\kappa}_2} x_2). \tag{2.1}$$

Then obviously

$$\varphi_{\tilde{\kappa}}(\delta_r^{\tilde{\kappa}}(x_1, x_2)) = r\varphi_{\tilde{\kappa}}(x_1, x_2)$$

holds true for every $r > 0$ and $(x_1, x_2) \in \mathbb{R}^2$, i.e. $\varphi_{\tilde{\kappa}}$ is $\tilde{\kappa}$-homogeneous of degree one. Observe that $\varphi_{\tilde{\kappa}}$ is not a monomial and that no point of the Taylor support of φ lies below the line $L_{\tilde{\kappa}}$. Following [21] we write

$$\varphi = \varphi_{\tilde{\kappa}} + R_{\tilde{\kappa}}, \tag{2.2}$$

where $R_{\tilde{\kappa}} = \varphi - \varphi_{\tilde{\kappa}}$ is the analytic remainder term of φ consisting only of terms of higher $\tilde{\kappa}$-degree. More precisely, every point (α, β) in the Taylor support of $R_{\tilde{\kappa}}$ lies above the line $L_{\tilde{\kappa}}$, i.e. $\tilde{\kappa}_1\alpha + \tilde{\kappa}_2\beta > 1$. Using Lemma 2.2.5 we see that

$$\varphi_{\tilde{\kappa}}(x_1, x_2) = C x_1^A x_2^B \prod_{j=1}^{M} (x_2^q - \lambda_j x_1^p)^{n_j}. \tag{2.3}$$

We put $n = \sum\limits_{j=1}^{M} n_j$. From the representation (2.3) we can read off that

$$\mathfrak{E} = [(A, B + qn), (A + pn, B)].$$

Denote by $\mathcal{H}$ the right half-plane

$$\mathcal{H} = \left\{(x_1, x_2) \in \mathbb{R}^2 : \; x_1 > 0\right\}.$$

Notice that the partial derivative $\partial_2\varphi_{\tilde{\kappa}}$ is $\tilde{\kappa}$-homogeneous of degree $1 - \tilde{\kappa}_2$ and that $\frac{\tilde{\kappa}_2}{\tilde{\kappa}_1} = \frac{p}{q}$.

Lemma 2.3.1. *Assume that $\partial_2\varphi_{\tilde{\kappa}}$ has a zero in $\mathcal{H}$. Because of the $\tilde{\kappa}$-homogeneity the polynomial $\partial_2\varphi_{\tilde{\kappa}}$ vanishes in $\mathcal{H}$ along finitely many curves*

$$\mathcal{C}_{d_i} = \left\{(r, d_i r^{\frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}}) : \; r > 0\right\},$$

where d_i are real numbers. Let $d = d_i$ be one of those numbers. Consider the change of variables on $\mathcal{H}$

$$\zeta(x_1, x_2) = (x_1, x_2 + d x_1^{\frac{p}{q}}).$$

Set $\tilde{\psi} = \varphi \circ \zeta$. Then $\tilde{\psi}$ is analytic in the variables $x_1^{\frac{1}{q}}$ and x_2, i.e. there exists an analytic function ψ such that $\tilde{\psi}(x_1, x_2) = \psi(x_1^{\frac{1}{q}}, x_2)$. Moreover, there exists $(P_1, P_2) \in \mathbb{N}_0^2$, $B + qn \geq P_2$, such that the Newton diagram $\mathcal{N}_d(\psi)$ contains the face $[(qA, B + qn), (P_1, P_2)]$ lying on the line

$$L_\kappa = \{(t_1, t_2) : \; \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

where

$$\kappa = (\kappa_1, \kappa_2) = \left(\frac{\tilde{\kappa}_1}{q}, \tilde{\kappa}_2\right) = \left(\frac{1}{m}, \frac{p}{m}\right).$$

Then either $P_2 = 0$ or $P_2 \geq 2$. $P_2 \geq 2$ if and only if $\varphi_{\tilde{\kappa}}(r, dr^{\frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}}) = 0$ for every $r > 0$, i.e. $\varphi_{\tilde{\kappa}}$ vanishes identically along $\mathcal{C}_d$. Furthermore, the face $[(qA, B+qn), (P_1, P_2)]$ of $\mathcal{N}(\psi)$ is a vertex if and only if $\varphi_{\tilde{\kappa}}(x_1, x_2) = Cx_1^A(x_2 - dx_1^p)^n$. In any case we have

$$[(qA, B+qn), (P_1, P_2)] \cap \{(t_1, 1) : \; t_1 \in \mathbb{R}\} \cap \mathcal{T}(\psi) = \emptyset. \tag{2.4}$$

Proof. Combining (2.2) and (2.3) we decompose φ in

$$\varphi(x_1, x_2) = \underbrace{Cx_1^A x_2^B \prod_{j=1}^{M} (x_2^q - \lambda_j x_1^p)^{n_j}}_{\varphi_{\tilde{\kappa}}} + \underbrace{\sum_{\tilde{\kappa}_1\alpha + \tilde{\kappa}_2\beta > 1} c_{\alpha,\beta} x_1^\alpha x_2^\beta}_{R_{\tilde{\kappa}}}.$$

With $\tilde{x}_1 = x_1^{\frac{1}{q}}$ we get

$$R_{\tilde{\kappa}} \circ \zeta(x_1, x_2) = R_{\tilde{\kappa}}(x_1, x_2 + dx_1^{\frac{p}{q}}) = R_{\tilde{\kappa}}(\tilde{x}_1^q, x_2 + d\tilde{x}_1^p) =: R_\kappa(\tilde{x}_1, x_2).$$

The function R_κ is obviously analytic in $(\tilde{x}_1, x_2)$ and $\mathcal{T}(R_\kappa)$ lies above the line L_κ. We get

$$\tilde{\psi}(x_1, x_2) = \varphi_{\tilde{\kappa}}(x_1, x_2 + dx_1^{\frac{p}{q}}) + R_\kappa(\tilde{x}_1, x_2).$$

Observe that the polynomial

$$\psi_\kappa(\tilde{x}_1, x_2) := \varphi_{\tilde{\kappa}}(\tilde{x}_1^q, x_2 + d\tilde{x}_1^p) = \varphi_{\tilde{\kappa}}(x_1, x_2 + dx_1^{\frac{p}{q}})$$

is κ-homogeneous of degree one in the variables $(\tilde{x}_1, x_2)$ and therefore its Taylor support lies on the line L_κ. We shall now analyze $\mathcal{N}_d(\psi_\kappa)$. Two mutually excluding cases can occur:

(i) $\varphi_{\tilde{\kappa}}$ vanishes identically along the curve $\mathcal{C}_d$;

(ii) $\varphi_{\tilde{\kappa}}$ does not vanish along the curve $\mathcal{C}_d$.

We get

$$\psi_\kappa(\tilde{x}_1, x_2) = C\tilde{x}_1^{qA}(x_2 + d\tilde{x}_1^p)^B \prod_{j=1}^{M} ((x_2 + d\tilde{x}_1^p)^q - \lambda_j \tilde{x}_1^{pq})^{n_j}.$$

We see that the left upper point of $\mathcal{N}_d(\psi_\kappa)$ is always $(qA, B+qn)$. First, assume that (i) holds. Therefore $\varphi_{\tilde{\kappa}}$ and $\partial_2\varphi_{\tilde{\kappa}}$ both vanish along the curve $\mathcal{C}_d$. Therefore the multiplicity of this zero of $\varphi_{\tilde{\kappa}}$ is then obviously at least two. If $d = 0$, then $B \geq 2$ and ζ is just the identity. In this case we conclude

$$\mathcal{N}_d(\psi_\kappa) = [(qA, B+qn), (qA+pqn, B)],$$

and since $n \geq 1$, we get $B < B + qn$.
Now, assume that $0 \neq d = \lambda_l^{\frac{1}{q}}$ for some $l \in \{1, \dots, M\}$. The highest power of $\tilde{x}_1$ is then clearly $qA + pB + pq(n - n_l) + p(q-1)n_l$. We get

$$(P_1, P_2) = (qA + pB + pq(n - n_l) + p(q-1)n_l, n_l).$$

The important observation now is that $2 \leq P_2 = n_l < B + qn$, unless $B = 0$, $q = 1$ and $n = n_l$. But this clearly implies $\varphi_{\tilde{\kappa}}(x_1, x_2) = Cx_1^A(x_2 - dx_1^p)^n$.
Now assume that (ii) holds. Again in the case $d = 0$ we conclude that ζ is just the identity. Therefore $B = 0$ and $\mathcal{N}_d(\psi_\kappa) = [(qA, B+qn), (qA+pqn, B)] = [(qA, qn), (qA+pqn, 0)]$. If $d \neq 0$, then we see that, since $d \notin \{\lambda_1^{\frac{1}{q}}, \dots, \lambda_n^{\frac{1}{q}}\}$, the highest power of $\tilde{x}_1$ is given by $qA + pB + pqn$. This implies $(P_1, P_2) = (qA + pB + pqn, 0)$. The identity (2.4) is clear if $P_2 \geq 2$. If $P_2 = 0$, then the identity (2.4) follows from

$$\partial_2\psi_\kappa(\tilde{x}_1, x_2)\big|_{x_2=0} = \partial_2\varphi_{\tilde{\kappa}}(\tilde{x}_1^q, x_2 + d\tilde{x}_1^p)\big|_{x_2=0} \equiv 0,$$

which implies $\psi_\kappa(\tilde{x}_1, x_2) = c_1\tilde{x}_1^{qA+pB+pqn} + x_2^\alpha Q(\tilde{x}_1, x_2)$ with a polynomial Q, $c_1 \neq 0$ and $\alpha \geq 2$. $\square$

3 Estimates for oscillatory integrals

In this chapter we provide several crucial uniform estimates for different classes of oscillatory integrals which will be needed to obtain the L^2-boundedness of the maximal operator in the regions near certain singularities. First, we shall recall two important lemmas, namely the van der Corput lemma and the method of stationary phase.

3.1 Van der Corput lemma

Lemma 3.1.1. *Let φ be a real-valued smooth function defined on (a,b), $-\infty < a < b < \infty$. If $\inf\limits_{x\in(a,b)} |\varphi^{(k)}(x)| \geq 1$ for some $k \geq 2$, then*

$$\left| \int_a^b e^{i\lambda\varphi(x)} dx \right| \leq c_k \lambda^{-\frac{1}{k}}$$

holds for every $\lambda > 0$ with a positive constant c_k independent of φ and (a,b).

The proof is easily done by induction on k. For the proof we refer the reader to [11], [43].

Corollary 3.1.2. *Two immediate consequences from the above lemma are:*

(i) If φ from the previous lemma satisfies $\inf\limits_{x\in(a,b)} |\varphi^{(k)}(x)| \geq c_0$ for some constant $c_0 > 0$, then

$$\left| \int_a^b e^{i\lambda\varphi(x)} dx \right| \leq c_k c_0^{-\frac{1}{k}} \lambda^{-\frac{1}{k}}$$

holds true for every $\lambda > 0$.

(ii) Under the assumptions on φ in the previous lemma, we can conclude that for any $\psi \in C^1[a,b]$ the estimate

$$\left| \int_a^b e^{i\lambda\varphi(x)} \psi(x) dx \right| \le c_k \lambda^{-\frac{1}{k}} \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right)$$

holds true.

3.2 The method of stationary phase

Let $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be an open set and let $\Psi\colon U \times V \longrightarrow \mathbb{R}$ be a smooth real-valued function. Let $(\xi^0, x^0) \in U \times V$. Assume that

$$\nabla_x \Psi\left(\xi^0, x^0\right) = 0, \quad \det\left(\frac{\partial^2 \Psi}{\partial x_i \partial x_j} \left(\xi^0, x^0\right) \right)_{i,j=1,\ldots,n} \neq 0,$$

i.e. $\Psi\left(\xi^0, \cdot\right)$ has a non-degenerate critical point in x^0. By the implicit function theorem there exists a neighborhood U^0 of ξ^0, a neighborhood V^0 of x^0 and a smooth function

$$\tilde{x}\colon U^0 \longrightarrow V^0$$

such that

$$\left\{(\xi, x) \in U^0 \times V^0 :\ \nabla_x \Psi(\xi, x) = 0\right\} = \left\{(\xi, \tilde{x}(\xi)) :\ \xi \in U^0\right\}.$$

Consider the oscillatory integral

$$I(\lambda, \xi) = \int_{\mathbb{R}^n} e^{i\lambda\Psi(\xi,x)} \psi(\lambda, \xi, x) dx, \quad (\lambda, \xi) \in (0, \infty) \times U^0.$$

We assume that the smooth function $\psi\colon (0,\infty) \times U^0 \times V^0 \longrightarrow \mathbb{C}$ is a symbol of order 0 in λ, i.e. for all $(\alpha, \beta, \gamma) \in \mathbb{N}_0 \times \mathbb{N}_0^m \times \mathbb{N}_0^n$ there exists a constant $C_{\alpha,\beta,\gamma} \ge 0$ such that

$$\left| \left(\frac{\partial}{\partial \lambda}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta \left(\frac{\partial}{\partial x}\right)^\gamma \psi(\lambda, \xi, x) \right| \le C_{\alpha,\beta,\gamma} (1 + \lambda)^{-\alpha}$$

holds true for every $(\lambda, \xi, x) \in (0, \infty) \times U^0 \times V^0$. Furthermore, we assume that the function $\psi(\lambda, \cdot, \cdot)$ is compactly supported in $U^0 \times V^0$ uniformly in $\lambda > 0$, i.e. there is a compact set $K \subseteq U^0 \times V^0$ such that $\operatorname{supp} \psi(\lambda, \cdot, \cdot) \subseteq K$ for every $\lambda > 0$.

Theorem 3.2.1. *Suppose that Ψ and ψ are as above. Then for every $(\alpha, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^m$ there exists a constant $C_{\alpha,\beta} \geq 0$ such that*

$$\left| \left(\frac{\partial}{\partial \lambda}\right)^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\beta} \left(e^{-i\lambda\Psi(\xi,\tilde{x}(\xi))} I(\lambda, \xi)\right) \right| \leq C_{\alpha,\beta}(1+\lambda)^{-\frac{n}{2}-\alpha}$$

holds true for every $(\lambda, \xi) \in (0, \infty) \times U^0$, i.e. $(\lambda, \xi) \longmapsto e^{-i\lambda\Psi(\xi,\tilde{x}(\xi))} I(\lambda, \xi)$ is a symbol of order $-\frac{n}{2}$ in λ.

For the proof we refer the reader to [19], [33], [39].

3.3 Estimates for the oscillatory integral Λ

We shall study the uniform decay of the oscillatory integral

$$\Lambda(\lambda, \varepsilon, \sigma, \delta) = \int_{\mathbb{R}^2} e^{i\Phi(\lambda,x,\varepsilon,\sigma,\delta)} \chi(x, \varepsilon, \sigma, \delta) dx, \quad \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3,$$

with the smooth phase function

$$\Phi(\lambda, x, \varepsilon, \sigma, \delta) = \lambda_1 x_1 + \lambda_2 \varphi_1(x, \varepsilon, \delta) + \lambda_3 \varphi_2(x, \sigma, \delta).$$

The real-valued function Φ is defined on $\mathbb{R}^3 \times U \times I_1 \times I_2 \times V$, $U \subseteq \mathbb{R}^2$ open. V is a small neighborhood of the origin in $\mathbb{R}^k$ and $I_j = (-\alpha_j, \alpha_j)$ are small intervals centered at the origin. The amplitude function $\chi(\cdot, \varepsilon, \sigma, \delta)$ is compactly supported in U uniformly in $(\varepsilon, \sigma, \delta)$. To be more precise, this means that there is a compact set $\mathcal{C} \subset U$ such that the function $\chi(\cdot, \varepsilon, \sigma, \delta)$ is supported in $\mathcal{C}$ for every $(\varepsilon, \sigma, \delta) \in I_1 \times I_2 \times V \subseteq \mathbb{R}^{k+2}$. The function χ is smooth and we assume that all derivatives of χ in x are uniformly bounded in $(\varepsilon, \sigma, \delta)$, i.e.

$$\forall M \in \mathbb{N}_0 \quad \exists N \in \mathbb{N} \quad \forall (\varepsilon, \sigma, \delta) \in I_1 \times I_2 \times V : \quad \|\chi(\cdot, \varepsilon, \sigma, \delta)\|_{C^M(U)} \leq N.$$

The real-valued functions φ_i are also assumed to be smooth and to be of the form

$$\varphi_1(x_1, x_2, \varepsilon, \delta) = \varepsilon x_2 + f_1(x_1) + R_1(x_1, \delta), \quad (x_1, x_2, \varepsilon, \delta) \in U \times I_1 \times V,$$

$$\varphi_2(x_1, x_2, \sigma, \delta) = f_2(x_1, x_2, \sigma) + R_2(x_1, x_2, \delta), \quad (x_1, x_2, \sigma, \delta) \in U \times I_2 \times V.$$

The functions R_i are smooth perturbations in δ, i.e.

$$\forall M \in \mathbb{N}_0: \quad \|R_1(\cdot, \delta)\|_{C^M(\pi_1(U))} + \|R_2(\cdot, \cdot, \delta)\|_{C^M(U)} = o(1) \quad \text{as } |\delta| \longrightarrow 0,$$

where

$$\pi_1 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x_1, x_2) \longmapsto x_1$$

is the projection onto x_1-axis.

Example 3.3.1. Let $f \colon (-\epsilon, \epsilon) \to \mathbb{R}$ be a smooth function with $f(0) = 0$. Then

$$R(x, \delta) = f(\delta x), \quad x \in \left(\frac{1}{4}, 4\right), \ \delta \in \left(-\frac{\epsilon}{10}, \frac{\epsilon}{10}\right),$$

is a smooth perturbation in δ.

We are looking for uniform estimates of the form

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \leq \frac{C}{|\varepsilon|^{b_1} |\sigma|^{b_2} (1 + |\lambda|)^{\frac{1}{2} + b_3}}, \quad \lambda \in \mathbb{R}^3, \tag{3.1}$$

where $b_3 > 0$, $\varepsilon \neq 0$, $\sigma \neq 0$ and $(b_1, b_2) \in \mathbb{R}^2$. The constant C is required to be independent of $(\varepsilon, \sigma, \delta)$ and λ. We will not investigate sharp estimates for Λ.
First, we shall make some general observations. We can assume that $|\lambda| \geq 5$, since for every $|\lambda| < 5$ and every $a > 0$ we get the trivial estimate

$$\begin{aligned} |\Lambda(\lambda, \varepsilon, \sigma, \delta)| &\leq \|\chi(\cdot, \varepsilon, \sigma, \delta)\|_{L^1(\mathbb{R}^2)} \\ &\leq \|\chi(\cdot, \varepsilon, \sigma, \delta)\|_{L^1(\mathbb{R}^2)} \cdot \frac{6^{\frac{1}{2} + a}}{(1 + |\lambda|)^{\frac{1}{2} + a}}, \end{aligned}$$

which is better than all other results stated below.
We can also always assume that $|\lambda_1| \lesssim |\lambda_2| + |\lambda_3|$, since in the case $|\lambda_1| \gg |\lambda_2| + |\lambda_3|$ we

can argue applying integration by parts and obtain the better estimate

$$|\Lambda(\lambda,\varepsilon,\sigma,\delta)| \lesssim |\lambda|^{-N}$$

for every $N \in \mathbb{N}$.

The uniform estimate of the oscillatory integral Λ depends on the curvature of the functions f_i and on the ratio $\frac{|\sigma|}{|\varepsilon|}$. We shall state some local results in the form that if for some point $x^0 = (x_1^0, x_2^0) \in \mathcal{C}$ there are certain non-vanishing conditions on partial derivatives of f_i in x^0 or x_1^0, then there is a small neighborhood $U(x^0) \subset U$ of x^0, with the volume independent of $(\lambda,\varepsilon,\sigma,\delta)$, such that a certain estimate of the form (3.1) holds true for any amplitude $\chi(\cdot,\varepsilon,\sigma,\delta)$ which is compactly supported in $U(x^0)$. Of course, the constant C in (3.1) will also depend on certain Sobolev norms of $\chi(\cdot,\varepsilon,\sigma,\delta)$. The global results follow from covering the compact set $\mathcal{C}$ by finitely many such neighborhoods and applying a suitable smooth partition of unity. The uniform decay of Λ is then determined by the worst pointwise behavior. For this purpose we fix a point $x^0 \in U$. We remark that similar estimates were also considered in article [21].

In the next section we shall first discuss two situations where the oscillatory integral Λ is independent of σ, i.e. the phase function Φ is given by

$$\Phi(\lambda,x,\varepsilon,\delta) = \lambda_1 x_1 + \lambda_2(\varepsilon x_2 + f_1(x_1) + R_1(x_1,\delta)) + \lambda_3(f_2(x_1,x_2) + R_2(x_1,x_2,\delta)),$$

and

$$\chi(x,\varepsilon,\sigma,\delta) = \chi(x,\varepsilon,\delta).$$

3.3.1 Estimates for the oscillatory integral Λ of the first type

Lemma 3.3.2. *Assume that*

$$\min\left\{|\partial_2 f_2(x^0)|, |\partial_2^2 f_2(x^0)|\right\} > 0. \tag{3.2}$$

Furthermore, suppose there is some $m \geq 2$ such that

$$|f_1^{(m)}(x_1^0)| > 0. \tag{3.3}$$

Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{V} \subseteq I_1 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \delta)| \lesssim |\varepsilon|^{-1}(1+|\lambda|)^{-\frac{1}{2}-\frac{1}{m}} \tag{3.4}$$

holds true for all $(\lambda, \varepsilon, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{V}$.

Proof. Observe that in the cases $|\varepsilon\lambda_2| \ll |\lambda_3|$ or $|\lambda_3| \ll |\varepsilon\lambda_2|$ we can integrate by parts in x_2, because of (3.2), and obtain

$$|\Lambda(\lambda, \varepsilon, \delta)| \lesssim \frac{1}{|\varepsilon||\lambda|}.$$

Therefore in cases $|\varepsilon\lambda_2| \ll |\lambda_3|$, $|\lambda_3| \ll |\varepsilon\lambda_2|$ we obtain the even stronger estimate

$$|\Lambda(\lambda, \varepsilon, \delta)| \lesssim |\varepsilon|^{-1}(1+|\lambda|)^{-1}. \tag{3.5}$$

Assume therefore that $|\varepsilon\lambda_2| \sim |\lambda_3|$. Then clearly, $\varepsilon\lambda_2 \neq 0$ and for $s = \frac{\lambda_3}{\lambda_2\varepsilon}$ we have $|s| \sim 1$. More precisely, we can assume that s is contained in some small neighborhood $N(s^0)$ of $s^0 = -\frac{1}{\partial_2 f_2(x^0)}$, since otherwise integration by parts in x_2 yields (3.5). In particular, $|\lambda_2| \sim |\lambda|$. Rewrite Λ as

$$I(\mu, \lambda_2, r, s, \varepsilon, \delta) = \int\limits_{\mathbb{R}} e^{i\lambda_2(rx_1+f_1(x_1)+R_1(x_1,\delta))} \left(\int\limits_{\mathbb{R}} e^{i\mu\rho(x_1,\delta,s,x_2)} \chi(x,\varepsilon,\delta) dx_2 \right) dx_1, \tag{3.6}$$

where

$$\rho(x_1, \delta, s, x_2) = x_2 + sf_2(x_1, x_2) + sR_2(x_1, x_2, \delta), \quad \mu = \lambda_2\varepsilon, \quad r = \frac{\lambda_1}{\lambda_2}.$$

Observe that $|r| \lesssim 1$. It is sufficient to show that there exist a neighborhood $U(x^0)$ of x^0, a neighborhood $\tilde{I}_1 \times \tilde{V}$ of the origin and a neighborhood $\tilde{N}(s^0)$ of s^0 such that for any smooth function $\chi(\cdot, \varepsilon, \delta)$ supported in $U(x^0)$ the estimate

$$|I(\mu, \lambda_2, r, s, \varepsilon, \delta)| \lesssim \frac{1}{|\mu|^{\frac{1}{2}}|\lambda_2|^{\frac{1}{m}}}$$

holds true for any $|\lambda_2| \gg 1$, $|r| \lesssim 1$, $(s, \varepsilon, \delta) \in \tilde{N}(s^0) \times \tilde{I}_1 \setminus \{0\} \times \tilde{V}$ and $\frac{|\mu|}{|\lambda_2|} = |\varepsilon|$. Using assumption (3.2) we see that the function $\rho(x_1^0, 0, s^0, \cdot)$ has a non-degenerate critical point in x_2^0. By the implicit function theorem, there exist a neighborhood

$$U_1(x_1^0) \times \tilde{V} \times \tilde{N}(s^0) \subseteq \pi_1(U) \times V \times N(s^0)$$

of $(x_1^0, 0, s^0)$, a neighborhood $U_2(x_2^0) \subseteq \pi_2(U)$ of x_2^0 and a smooth function $y = y(x_1, \delta, s)$ defined on $U_1(x_1^0) \times \tilde{V} \times \tilde{N}(s^0)$ with values in $U_2(x_2^0)$ such that

$$\partial_{x_2}\rho(x_1, \delta, s, y(x_1, \delta, s)) = 0 \quad \text{for every } (x_1, \delta, s) \in U_1(x_1^0) \times \tilde{V} \times \tilde{N}(s^0).$$

Assume that $\chi(\cdot, \varepsilon, \delta)$ is supported in $U_1(x_1^0) \times U_2(x_2^0)$.
Notice that $U_1(x_1^0) \times U_2(x_2^0) \times \tilde{V} \times \tilde{N}(s^0)$ is independent of $(\mu, \lambda_2, \varepsilon)$. Applying Theorem 3.2.1 to the inner integral, we see that (3.6) can be written as

$$I(\mu, \lambda_2, r, s, \varepsilon, \delta) = \int\limits_{\mathbb{R}} e^{i\lambda_2(rx_1 + f_1(x_1) + R_1(x_1, \delta))} e^{i\mu\rho(x_1, \delta, s, y(x_1, \delta, s))} F(\mu, x_1, s, \varepsilon, \delta) dx_1,$$

where F is a symbol of order $-\frac{1}{2}$ in μ. In particular, each partial derivative of any order $K \in \mathbb{N}_0$ satisfies

$$\left|\partial_{x_1}^K F(\mu, x_1, s, \varepsilon, \delta)\right| \leq C_K |\mu|^{-\frac{1}{2}}$$

uniformly in $(x_1, \varepsilon, s, \delta)$. First, observe that

$$\begin{aligned} & \lambda_2(rx_1 + f_1(x_1) + R_1(x_1, \delta)) + \mu\rho(x_1, \delta, s, y(x_1, \delta, s)) \\ = \;& \lambda_2(rx_1 + f_1(x_1) + R_1(x_1, \delta) + \varepsilon\rho(x_1, \delta, s, y(x_1, \delta, s))), \end{aligned}$$

and that

$$\partial_{x_1} y(x_1, \delta, s) = -\frac{s\partial_1\partial_2 f_2(x_1, y(x_1, \delta, s)) + s\partial_1\partial_2 R_2(x_1, y(x_1, \delta, s), \delta)}{s\partial_2^2 f_2(x_1, y(x_1, \delta, s)) + s\partial_2^2 R_2(x_1, y(x_1, \delta, s), \delta)} = \mathcal{O}(1),$$

since $|s| \sim 1$, $\partial_2^2 f_2(x^0) \neq 0$ and R_2 is a perturbation term. An induction shows that $\partial_{x_1}^K y(x_1, \delta, s) = \mathcal{O}(1)$ for every $K \in \mathbb{N}$ on $U_1(x_1^0) \times \tilde{V} \times \tilde{N}(s^0)$ and this also implies $\partial_{x_1}^K \rho(x_1, \delta, s, y(x_1, s, \delta)) = \mathcal{O}(1)$. By assumption (3.3), and if we assume (ε, δ) and $U_1(x_1^0)$ to be small enough, we conclude

$$\left|\partial_{x_1}^m (rx_1 + f_1(x_1) + R_1(x_1, \delta) + \varepsilon\rho(x_1, \delta, s, y(x_1, \delta, s)))\right| \gtrsim 1$$

uniformly in ε, δ, s and $x_1 \in U_1(x_1^0)$. With the van der Corput lemma we can eventually conclude that

$$|I(\mu, \lambda_2, r, s, \varepsilon, \delta)| \lesssim \frac{1}{|\mu|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{m}}} \lesssim \frac{1}{|\varepsilon|^{\frac{1}{2}} (1 + |\lambda|)^{\frac{1}{2} + \frac{1}{m}}}. \tag{3.7}$$

□

Remark 3.3.3. *As already mentioned, we will not determine the asymptotic behavior of Λ. For example, if we distinguish two additional cases $|\lambda_2| \ll |\lambda_3|$ and $|\lambda_3| \ll |\lambda_2|$, we can even show that the estimate (3.4) can be improved to (3.7). Indeed, the estimate (3.7) is sharp. This is seen if we set*

$$f_1(x_1) = x_1 + x_1^m, \quad f_2(x_1, x_2) = x_2 - x_2^2, \quad R_1 \equiv 0 \equiv R_2.$$

If χ is supported in a small neighborhood of the origin, then along the line $1 \ll \lambda_2 = -\lambda_1$, $\lambda_3 = -\varepsilon \lambda_2$, $\varepsilon > 0$, the oscillatory integral Λ satisfies

$$|\Lambda(\lambda, \varepsilon, \delta)| \sim \lambda_2^{-\frac{1}{2} - \frac{1}{m}} \varepsilon^{-\frac{1}{2}} \sim |\lambda|^{-\frac{1}{2} - \frac{1}{m}} \varepsilon^{-\frac{1}{2}}.$$

We remark that the second part of the proof of the next lemma also follows from Proposition 5.2 in [21].

Lemma 3.3.4. *Assume that*

$$\left|\partial_2 f_2(x^0)\right| > 0, \tag{3.8}$$

and that

$$\min\left\{|f_1'(x_1^0)|, |f_1''(x_1^0)|\right\} > 0. \tag{3.9}$$

Furthermore, suppose there is some $m \geq 2$ such that

$$\left|\partial_2^m f_2(x^0)\right| > 0. \tag{3.10}$$

Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{V} \subseteq I_1 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \delta)| \lesssim |\varepsilon|^{-1} (1 + |\lambda|)^{-\frac{1}{2} - \frac{1}{m}} \tag{3.11}$$

holds true for all $(\lambda, \varepsilon, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{V}$.

Proof. Since $\partial_2 f_2(x^0) \neq 0$, we can assume that $|\varepsilon\lambda_2| \sim |\lambda_3|$, because otherwise integration by parts in x_2 yields stronger estimate (3.5). Notice that we can again assume that the fraction $\beta = \frac{\lambda_3}{\varepsilon\lambda_2}$ lies in some small neighborhood $N(\beta^0)$ of $\beta^0 = -\frac{1}{\partial_2 f_2(x^0)}$. In particular, we have $|\lambda| \sim |\lambda_2|$. Thus we rewrite Λ as

$$I(\lambda_2, s, \beta, \varepsilon, \delta) = \int_{\mathbb{R}} e^{i\lambda_2 \varepsilon x_2} \left(\int_{\mathbb{R}} e^{i\lambda_2 \rho(x_1, x_2, s, \beta, \varepsilon, \delta)} \chi(x, \varepsilon, \delta) dx_1 \right) dx_2,$$

where

$$\rho(x_1, x_2, s, \beta, \varepsilon, \delta) = sx_1 + f_1(x_1) + R_1(x_1, \delta) + \varepsilon\beta f_2(x_1, x_2) + \varepsilon\beta R_2(x_1, x_2, \delta), \quad s = \frac{\lambda_1}{\lambda_2}.$$

Notice that $|s| \lesssim 1$. Furthermore, we can assume that s lies in some small neighborhood $N(s^0)$ of $s^0 = -f_1'(x_1^0) \neq 0$, since otherwise we make use of (3.9), integrate by parts in x_1 and obtain the better decay $(1 + |\lambda_2|)^{-1}$. We shall prove that there exists a neighborhood $U(x^0)$ of x^0, a neighborhood $\tilde{I}_1 \times \tilde{V} \subseteq I_1 \times V$ of the origin and a neighborhood $\tilde{N}(s^0) \times \tilde{N}(\beta^0)$ of (s^0, β^0) such that for any smooth function $\chi(\cdot, \varepsilon, \delta)$ supported in $U(x^0)$ the estimate

$$|I(\lambda_2, s, \beta, \varepsilon, \delta)| \lesssim \frac{1}{|\varepsilon|^{\frac{1}{m}} |\lambda_2|^{\frac{1}{2} + \frac{1}{m}}}$$

holds true for any $|\lambda_2| \gg 1$ and $(s, \beta, \varepsilon, \delta) \in \tilde{N}(s^0) \times \tilde{N}(\beta^0) \times \tilde{I}_1 \setminus \{0\} \times \tilde{V}$. Using assumption (3.9), we see that $\rho(\cdot, x_2^0, s^0, \beta^0, 0, 0)$ has a non-degenerate critical point in x_1^0. By the implicit function theorem there is a smooth function $\tilde{x} = \tilde{x}(x_2, s, \beta, \varepsilon, \delta)$ defined on some neighborhood $U_2(x_2^0) \times \tilde{N}(s^0) \times \tilde{N}(\beta^0) \times \tilde{I}_1 \times \tilde{V}$ of $(x_2^0, s^0, \beta^0, 0, 0)$ with values in $U_1(x_1^0)$ such that

$$s + f_1'(\tilde{x}) + \partial_1 R_1(\tilde{x}, \delta) + \varepsilon\beta\partial_1 f_2(\tilde{x}, x_2) + \varepsilon\beta\partial_1 R_2(\tilde{x}, x_2, \delta) = 0 \tag{3.12}$$

for any $(x_2, s, \beta, \varepsilon, \delta) \in U_2(x_2^0) \times \tilde{N}(s^0) \times \tilde{N}(\beta^0) \times \tilde{I}_1 \times \tilde{V}$. Next, we see that

$$\partial_{x_2} \tilde{x}(x_2, s, \beta, \varepsilon, \delta) = -\frac{\varepsilon\beta\partial_2\partial_1 f_2(\tilde{x}, x_2) + \varepsilon\beta\partial_2\partial_1 R_2(\tilde{x}, x_2, \delta)}{f_1''(\tilde{x}) + \partial_1^2 R_1(\tilde{x}, \delta) + \varepsilon\beta\partial_1^2 f_2(\tilde{x}, x_2) + \varepsilon\beta\partial_1^2 R_2(\tilde{x}, x_2, \delta)} = \mathcal{O}(\varepsilon).$$

By induction we see that $\partial_{x_2}^K \tilde{x}(x_2, s, \beta, \varepsilon, \delta) = \mathcal{O}(\varepsilon)$ for every $K \in \mathbb{N}$. Assume that the amplitude χ is supported in $U(x_1^0) \times U(x_2^0)$. Applying the method of stationary phase to the inner integral, we write

$$I(\lambda_2, s, \beta, \varepsilon, \delta) = \int_{\mathbb{R}} e^{i\lambda_2(\varepsilon x_2 + \rho(\tilde{x}(x_2,s,\beta,\varepsilon,\delta),s,\beta,\varepsilon,\delta))} F(\lambda_2, x_2, s, \beta, \varepsilon, \delta) dx_2,$$

where F is a symbol of order $-\frac{1}{2}$ in λ_2. Set $\gamma = \gamma(s, \beta, \varepsilon, \delta) = (s, \beta, \varepsilon, \delta)$. Notice that

$$\begin{aligned}
\rho(\tilde{x}(x_2,\gamma), x_2, \gamma) &= s\tilde{x}(x_2,\gamma) + f_1(\tilde{x}(x_2,\gamma)) \\
&\quad + R_1(\tilde{x}(x_2,\gamma),\delta) + \varepsilon\beta f_2(\tilde{x}(x_2,\gamma), x_2) \\
&\quad + \varepsilon\beta R_2(\tilde{x}(x_2,\gamma), x_2, \delta).
\end{aligned}$$

Using (3.12) we get

$$\begin{aligned}
\partial_{x_2}\rho(\tilde{x}(x_2,\gamma), x_2, \gamma) &= s\partial_{x_2}\tilde{x}(x_2,\gamma) + f_1'(\tilde{x}(x_2,\gamma))\partial_{x_2}\tilde{x}(x_2,\gamma) \\
&\quad + \partial_1 R_1(\tilde{x}(x_2,\gamma),\delta)\partial_{x_2}\tilde{x}(x_2,\gamma) + \varepsilon\beta\partial_1 f_2(\tilde{x}(x_2,\gamma), x_2)\partial_{x_2}\tilde{x}(x_2,\gamma) \\
&\quad + \varepsilon\beta\partial_2 f_2(\tilde{x}(x_2,\gamma), x_2) + \varepsilon\beta\partial_1 R_2(\tilde{x}(x_2,\gamma), x_2, \delta)\partial_{x_2}\tilde{x}(x_2,\gamma) \\
&\quad + \varepsilon\beta\partial_2 R_2(\tilde{x}(x_2,\gamma), x_2, \delta) \\
&= \varepsilon\beta\partial_2 f_2(\tilde{x}(x_2,\gamma), x_2) + \varepsilon\beta\partial_2 R_2(\tilde{x}(x_2,\gamma), x_2, \delta).
\end{aligned}$$

Observe that the function

$$\varepsilon\beta\partial_2 R_2(\tilde{x}(x_2,\gamma), x_2, \delta) = \varepsilon\beta\partial_2 R_2(\tilde{x}(x_2, s, \beta, \varepsilon, \delta), x_2, \delta) = \varepsilon\beta \cdot o(1)$$

for $|\delta| \longrightarrow 0$ in each C^M-norm with respect to x_2.
We conclude

$$\begin{aligned}
\partial_{x_2}^2\rho(\tilde{x}(x_2,\gamma), x_2, \gamma) &= \varepsilon\beta\partial_1\partial_2 f_2(\tilde{x}(x_2,\gamma), x_2, \gamma)\partial_{x_2}\tilde{x}(x_2,\gamma) \\
&\quad + \varepsilon\beta\partial_2^2 f_2(\tilde{x}(x_2,\gamma), x_2) + \varepsilon\beta \cdot o(1) \\
&= \mathcal{O}(\varepsilon^2)\partial_1\partial_2 f_2(\tilde{x}(x_2,\gamma), x_2) + \varepsilon\beta\partial_2^2 f_2(\tilde{x}(x_2,\gamma), x_2) + \varepsilon\beta \cdot o(1).
\end{aligned}$$

By induction it is easily seen that

$$\partial_{x_2}^K\rho(\tilde{x}(x_2,\gamma), x_2, \gamma) = \mathcal{O}(\varepsilon^2) + \varepsilon\beta\partial_2^K f_2(\tilde{x}(x_2,\gamma), x_2) + \varepsilon\beta \cdot o(1)$$

holds for every $K \geq 2$.

With the van der Corput lemma, using (3.10), we conclude for $\varepsilon \neq 0$

$$|I(\lambda_2, s, \beta, \varepsilon, \delta)| \lesssim |\varepsilon|^{-\frac{1}{m}} |\lambda_2|^{-\frac{1}{2}-\frac{1}{m}}$$

if $U_2(x_2^0) \times \tilde{N}(s^0) \times \tilde{N}(\beta^0) \times \tilde{I}_1 \times \tilde{V}$ is small enough. $\square$

3.3.2 Estimates for the oscillatory integral Λ of the second type

Recall that the phase function Φ is given by

$$\Phi(\lambda, x, \varepsilon, \sigma, \delta) = \lambda_1 x_1 + \lambda_2 \varphi_1(x, \varepsilon, \delta) + \lambda_3 \varphi_2(x, \sigma, \delta).$$

In order to simplify the assumptions, we shall assume that $\pi_1(\operatorname{supp} \chi(\cdot, \varepsilon, \sigma, \delta))$ is contained in some compact interval $J \subseteq (0, \infty)$ uniformly in $(\varepsilon, \sigma, \delta) \in I_1 \times I_2 \times V$. Assume that φ_i are smooth real-valued functions given by

$$\varphi_1(x_1, x_2, \varepsilon, \delta) = \varepsilon x_2 + c_1 x_1^{\gamma_1} + R_1(x_1, \delta),$$

$$\varphi_2(x_1, x_2, \sigma, \delta) = c_2 x_1^{\gamma_2} + R_3(x_1, \delta) + \sigma g(x_1, x_2) + \sigma R_2(x_1, x_2, \delta),$$

where $c_i \in \mathbb{R} \setminus \{0\}$. Furthermore, we assume that R_3 is also a perturbation term in δ. The exponents γ_i are positive real numbers and satisfy

$$1 \neq \gamma_1 < \gamma_2 \neq 1.$$

As already mentioned the estimates of the oscillatory integral Λ will depend on certain curvature conditions and on the size of the ratio $\frac{|\sigma|}{|\varepsilon|}$. The next proposition is a well known result which is also easy to check.

Proposition 3.3.5. *Let $P(x_1) = \alpha_1 x_1^{\gamma_1} + \alpha_2 x_1^{\gamma_2}$, $\alpha_i \in \mathbb{R} \setminus \{0\}$. Then there exists a constant $C = C(J, \gamma_1, \gamma_2, |\alpha_1|, |\alpha_2|) > 0$ such that*

$$\inf_i \inf_{x_1 \in J} \left| \frac{d}{dx_1} \alpha_i x_1^{\gamma_i} \right| \geq C, \quad \inf_i \inf_{x_1 \in J} \left| \frac{d^2}{dx_1^2} \alpha_i x_1^{\gamma_i} \right| \geq C,$$

$$\inf_{x_1 \in J} |P''(x_1)| + |P'''(x_1)| \geq C.$$

In the following we will discuss several cases. The first observation is that we obtain a good uniform decay in the case $|\lambda_2| \ll |\lambda_3|$, and if in addition g is of finite type $m \geq 2$ in x_2 at x^0, i.e. $\partial_2^m g(x^0) \neq 0$.

Lemma 3.3.6. *Assume there exists $m \geq 2$ such that*

$$\left|\partial_2^m g(x^0)\right| > 0.$$

Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{I}_2 \times \tilde{V} \subseteq I_1 \times I_2 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\sigma|^{\frac{1}{m}}(1 + |\lambda|)^{\frac{1}{2}+\frac{1}{m}}} \tag{3.13}$$

holds true for all $(\lambda, \varepsilon, \sigma, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \times \tilde{I}_2 \setminus \{0\} \times \tilde{V}$ and $|\lambda_2| \ll |\lambda_3|$.

Proof. The arguments are the same as in the proof of Lemma 3.3.4. In fact, we can assume that $|\lambda_1| \sim |\lambda_3|$, since otherwise we can integrate by parts in x_1. By the method of stationary phase in x_1 with combination of van der Corput lemma in x_2 we obtain

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\sigma|^{\frac{1}{m}}|\lambda_3|^{\frac{1}{2}+\frac{1}{m}}}$$

for $0 < |\sigma|$ and $\frac{|\lambda_2|}{|\lambda_3|}$ small enough. $\square$

In the following we can always assume that $\frac{|\lambda_2|}{|\lambda_3|} \geq \varepsilon_0$ for some small positive constant ε_0.

Lemma 3.3.7. *Assume there exists $m \geq 2$ such that*

$$\left|\partial_2^m g(x^0)\right| > 0.$$

Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{I}_2 \times \tilde{V} \subseteq I_1 \times I_2 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{\min\{|\sigma|^{\frac{1}{m}}, |\varepsilon|\}(1 + |\lambda|)^{\frac{1}{2}+\frac{1}{m}}} \tag{3.14}$$

holds true for all $(\lambda, \varepsilon, \sigma, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{I}_2 \setminus \{0\} \times \tilde{V}$ and $|\sigma| \ll |\varepsilon|$. The estimate (3.14) is also valid if $|\varepsilon| \sim |\sigma|$ if in addition $\partial_2 g(x^0) = 0$ and $U(x^0)$ is small enough.

Proof. First, we discuss the special case when the quotient $\frac{|\sigma|}{|\varepsilon|}$ is sufficiently small (now also with respect to ε_0). Recall first that we are only left with the case $|\lambda| \sim |\lambda_2|$. We write

$$\Lambda(\lambda, \varepsilon, \sigma, \delta) = \int\limits_{\mathbb{R}} e^{i\lambda_2 \rho_1(x_1,\lambda,\delta)} \left(\int\limits_{\mathbb{R}} e^{i\lambda_2 \varepsilon \rho_2(x_1,x_2,\lambda_2,\lambda_3,\varepsilon,\sigma,\delta)} \chi(x, \varepsilon, \sigma, \delta) dx_2 \right) dx_1,$$

where

$$\rho_1(x_1, \lambda, \delta) = \frac{\lambda_1}{\lambda_2} x_1 + c_1 x_1^{\gamma_1} + R_1(x_1, \delta) + \frac{\lambda_3}{\lambda_2} c_2 x_1^{\gamma_2} + \frac{\lambda_3}{\lambda_2} R_3(x_1, \delta),$$

$$\rho_2(x_1, x_2, \lambda_2, \lambda_3, \varepsilon, \sigma, \delta) = x_2 + \frac{\lambda_3 \sigma}{\lambda_2 \varepsilon} g(x_1, x_2) + \frac{\lambda_3 \sigma}{\lambda_2 \varepsilon} R_2(x_1, x_2, \delta).$$

Thus we see that in the case $\frac{|\sigma|}{|\varepsilon|} \ll 1$ we can integrate by parts in x_2 and conclude that

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\varepsilon \lambda_2|} \sim \frac{1}{|\varepsilon|(1 + |\lambda|)} \lesssim \frac{1}{\min\{|\varepsilon|, |\sigma|^{\frac{1}{m}}\}(1 + |\lambda|)^{\frac{1}{2}+\frac{1}{m}}}.$$

The same argument also applies if $|\sigma| \sim |\varepsilon|$, $\partial_2 g(x^0) = 0$ and the function $\chi(\cdot, \varepsilon, \sigma, \delta)$ is supported in a sufficiently small neigborhood of x^0. ☐

Remark 3.3.8. *Later in the applications the parameters ε, σ will be of the dyadic form, $(\sigma, \varepsilon) = (2^{-ja}, 2^{-jb})$, a, $b > 0$, for some large positive integer j. In particular, $\frac{|\sigma|}{|\varepsilon|} = 2^{-j(a-b)}$. If $a > b$, and j can be chosen sufficiently large with respect to all other quantities, the quotient is arbitrary small with respect to all other quantities. In particular, one of the three cases $|\sigma| = |\varepsilon|$, $|\sigma| \ll |\varepsilon|$, $|\sigma| \gg |\varepsilon|$ always holds true.*

In the next lemma we assume that $\min\{|\partial_2 g(x^0)|, |\partial_2^2 g(x^0)|\} > 0$ and obtain a uniform global estimate of Λ in all variables.

Lemma 3.3.9. *Assume that*

$$\min\left\{\left|\partial_2 g(x^0)\right|, \left|\partial_2^2 g(x^0)\right|\right\} > 0. \tag{3.15}$$

Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{I}_2 \times \tilde{V} \subseteq I_1 \times I_2 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{\min\{|\varepsilon|, |\sigma|^{\frac{1}{2}}\}(1 + |\lambda|)^{\frac{5}{6}}} \tag{3.16}$$

holds true for all $(\lambda, \varepsilon, \sigma, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{I}_2 \setminus \{0\} \times \tilde{V}$.

Proof. Recall that we only have to consider the case $|\lambda_1| \lesssim |\lambda_2| + |\lambda_3|$, $|\lambda| \geq 5$. In view of Lemma 3.3.6 we can assume that $|\lambda_3| \lesssim |\lambda_2|$, since otherwise $|\lambda_3| \gg |\lambda_2|$ implies the even better estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\lambda_3||\sigma\lambda_3|^{\frac{1}{2}}}.$$

In particular, $|\lambda| \sim |\lambda_2|$. We are thus left with three different cases.

Case 1: $|\lambda_2\varepsilon| \ll |\lambda_3\sigma|$

Then we can integrate by parts in x_2 and obtain the stronger estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim |\sigma\lambda_3|^{-1} \lesssim |\varepsilon\lambda_2|^{-1} \sim \frac{1}{|\varepsilon|(1+|\lambda|)}.$$

This implies (3.16).

Case 2: $|\lambda_3\sigma| \ll |\lambda_2\varepsilon|$

Again using integration by parts we conclude

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim |\lambda_2\varepsilon|^{-1} \sim \frac{1}{|\varepsilon|(1+|\lambda|)}. \tag{3.17}$$

Thus we are left with the last case.

Case 3: $|\lambda_2\varepsilon| \sim |\lambda_3\sigma|$

Set $\beta_1 = \frac{\lambda_1}{\lambda_2}$, $\beta_2 = \frac{\lambda_3}{\lambda_2}$ and $\beta_3 = \frac{\lambda_3\sigma}{\lambda_2\varepsilon}$. Observe that $\max\{|\beta_1|, |\beta_2|\} \lesssim 1$ and $|\beta_3| \sim 1$. We rewrite Λ as

$$I(\lambda_2, \mu, \beta, \varepsilon, \sigma, \delta) = \int\limits_{\mathbb{R}} e^{i\lambda_2\rho_1(x_1,\beta_1,\beta_2,\delta)} \left(\int\limits_{\mathbb{R}} e^{i\mu\rho_2(x_1,x_2,\beta_3,\delta)} \chi(x, \varepsilon, \sigma, \delta) dx_2 \right) dx_1,$$

where

$$\beta = (\beta_1, \beta_2, \beta_3), \quad \mu = \lambda_2\varepsilon,$$

$$\rho_1(x_1, \beta_1, \beta_2, \delta) = \beta_1 x_1 + c_1 x_1^{\gamma_1} + R_1(x_1, \delta) + \beta_2 c_2 x_1^{\gamma_2} + \beta_2 R_3(x_1, \delta),$$

$$\rho_2(x_1, x_2, \beta_3, \delta) = x_2 + \beta_3 g(x_1, x_2) + \beta_3 R_2(x_1, x_2, \delta).$$

The desired estimate follows if we prove that

$$|I(\lambda_2, \mu, \beta, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\mu|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{3}}}$$

holds true uniformly for $|\lambda_2| \gg 1$, $\mu = \lambda_2 \varepsilon$, $\varepsilon \neq 0$, $|\beta_3| \sim 1$, $(\varepsilon, \sigma, \delta)$ sufficiently small, and $|\beta_1| + |\beta_2| \lesssim 1$. We can assume that β_3 lies in some small neighborhood $N(\beta_3^0)$ of $\beta_3^0 = -\frac{1}{\partial_2 g(x^0)}$, since otherwise integration by parts yields (3.17). By assumption $\partial_2^2 g(x^0) \neq 0$. Therefore the function $\rho_2(x_1^0, \cdot, \beta_3^0, 0)$ has a non-degenerate critical point in x_2^0. Application of the method of stationary phase to the inner integral gives

$$I(\lambda_2, \mu, \beta, \varepsilon, \sigma, \delta) = \int\limits_{\mathbb{R}} e^{i\lambda_2(\rho_1(x_1, \beta_1, \beta_2, \delta) + \varepsilon \rho_2(x_1, y(x_1, \beta_3, \delta), \beta_3, \delta))} F(\mu, x_1, \beta_3, \varepsilon, \sigma, \delta) dx_1,$$

where y is a smooth function defined on a small neighborhood $U(x_1^0) \times \tilde{N}(\beta_3^0) \times \tilde{V}$ of $(x_1^0, \beta_3^0, 0)$ satisfying $\partial_2 \rho_2(x_1, y(x_1, \beta_3, \delta), \beta_3, \delta) \equiv 0$. The function F is a symbol of order $-\frac{1}{2}$ in μ. Notice that each derivative $\partial_1^K y(x_1, \beta_3, \delta)$ is bounded, which in turn implies that $\partial_{x_1}^K \rho_2(x_1, y(x_1, \beta_3, \delta), \beta_3, \delta)$ is also bounded for every $K \in \mathbb{N}$. Observe that if $|\beta_2| \ll 1$, then in view of Proposition 3.3.5, we conclude that $\partial_{x_1}^2 G(x_1, \beta, \varepsilon, \delta)$ is bounded from below, where

$$G(x_1, \beta, \varepsilon, \delta) = \rho_1(x_1, \beta_1, \beta_2, \delta) + \varepsilon \rho_2(x_1, y(x_1, \beta_3, \delta), \beta_3, \delta),$$

provided ε and the neighborhood $U(x_1^0) \times \tilde{V}$ are chosen sufficiently small. In such a case we obtain the better estimate

$$|I(\lambda_2, \mu, \beta, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\mu|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{2}}}.$$

Therefore we can assume that $|\beta_2| \sim 1$. Then if ε and $\tilde{V}$ are sufficiently small we conclude from Proposition 3.3.5 that

$$\left|\partial_{x_1}^2 G(x_1, \beta, \varepsilon, \delta)\right| + \left|\partial_{x_1}^3 G(x_1, \beta, \varepsilon, \delta)\right|$$

is uniformly bounded from below. By a more general van der Corput estimate (formulated by J. E. Björk, cf. [9]) we conclude that

$$|I(\lambda_2, \mu, \beta, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\mu|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{3}}}.$$

$\square$

An immediate consequence of the proof of Lemma 3.3.9 is the next corollary.

Corollary 3.3.10. *Assume that g satisfies the assumptions (3.15). Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{I}_2 \times \tilde{V} \subseteq I_1 \times I_2 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ and for every $m \geq 2$ the estimate*

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{\min\{|\varepsilon|, |\sigma|^{\frac{1}{m}}\}(1+|\lambda|)^{\min\{\frac{5}{6}, \frac{1}{2}+\frac{1}{m}\}}} \tag{3.18}$$

holds true for all $(\lambda, \varepsilon, \sigma, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{I}_2 \setminus \{0\} \times \tilde{V}$.

Proof. In fact, this is easily seen, since in the case $|\lambda_3| \gg |\lambda_2|$ we can also assume that $|\lambda_3 \sigma| \geq 1$, because otherwise the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\lambda_3|^{\frac{1}{2}}} \lesssim \frac{1}{|\lambda_3|^{\frac{1}{2}} |\sigma \lambda_3|^{\frac{1}{m}}} \lesssim \frac{1}{\min\{|\varepsilon|, |\sigma|^{\frac{1}{m}}\}(1+|\lambda|)^{\min\{\frac{5}{6}, \frac{1}{2}+\frac{1}{m}\}}}$$

holds true. On the other hand, if $|\lambda_3 \sigma| \geq 1$, then the proven estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\lambda_3|^{\frac{1}{2}} |\lambda_3 \sigma|^{\frac{1}{2}}}$$

easily implies

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\lambda_3|^{\frac{1}{2}} |\lambda_3 \sigma|^{\frac{1}{m}}} \lesssim \frac{1}{\min\{|\varepsilon|, |\sigma|^{\frac{1}{m}}\}(1+|\lambda|)^{\min\{\frac{5}{6}, \frac{1}{2}+\frac{1}{m}\}}}.$$

All other cases easily imply (3.18). $\square$

In the next lemma we drop the additional assumption $\partial_2^2 g(x^0) \neq 0$ and obtain a global estimate in λ for $|\sigma| \gg |\varepsilon|$.

Lemma 3.3.11. *Assume that there exists $m \geq 2$ such that*

$$\min\left\{\left|\partial_2 g(x^0)\right|, \left|\partial_2^m g(x^0)\right|\right\} > 0. \tag{3.19}$$

Then there exist a neighborhood $U(x^0)$ of x^0 and a neighborhood $\tilde{I}_1 \times \tilde{I}_2 \times \tilde{V} \subseteq I_1 \times I_2 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{\min\{|\varepsilon|, |\sigma|^{\frac{1}{m}}\}(1+|\lambda|)^{\frac{1}{2}+\frac{1}{m}}} \tag{3.20}$$

holds true for all $(\lambda, \varepsilon, \sigma, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{I}_2 \setminus \{0\} \times \tilde{V}$ *with* $|\sigma| \gg |\varepsilon|$.

Proof. Assume that $|\sigma| \gg |\varepsilon|$ and that ε, σ are sufficiently small. We can again assume that $|\lambda_3| \lesssim |\lambda_2|$ and therefore $|\lambda_2| \sim |\lambda|$. The cases $|\lambda_2\varepsilon| \ll |\lambda_3\sigma|$ and $|\lambda_3\sigma| \ll |\lambda_2\varepsilon|$ were dealt in the previous lemma by integration by parts using only the assumption that $\partial_2 g(x^0) \neq 0$. In both cases we get a better estimate, namely

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\varepsilon|(1+|\lambda|)}.$$

Thus we are only left with the case $|\lambda_2\varepsilon| \sim |\lambda_3\sigma|$. In particular, $|\lambda_2| \gg |\lambda_3|$, since by assumption $|\sigma| \gg |\varepsilon|$. In this case we rewrite Λ as

$$I(\lambda_2, \beta_1, \beta_2, s, \varepsilon, \delta) = \int\limits_{\mathbb{R}} e^{i\lambda_2 x_2} \left(\int\limits_{\mathbb{R}} e^{i\lambda_2(\rho(x_1,\beta_1,\beta_2\delta)+sg(x_1,x_2)+sR_3(x_1,x_2,\delta))} \chi(x,\varepsilon,\sigma,\delta)\,dx_1 \right) dx_2,$$

where

$$\beta_1 = \frac{\lambda_1}{\lambda_2}, \quad \beta_2 = \frac{\lambda_3}{\lambda_2}, \quad s = \frac{\lambda_3\sigma}{\lambda_2},$$
$$\rho(x_1, \beta_1, \beta_2, \delta) = \beta_1 x_1 + c_1 x_1^{\gamma_1} + \beta_2 c_2 x_1^{\gamma_2} + \beta_2 R_1(x_1, \delta).$$

Notice that $|s| \sim |\varepsilon|$ and that β_2 is small enough, provided $\frac{|\sigma|}{|\varepsilon|}$ is large. We can assume that β_1 lies in a small neighborhood of $\beta_1^0 = -c_1\gamma_1(x_1^0)^{\gamma_1-1}$, since otherwise integration by parts yields

$$|I(\lambda_2, \beta_1, \beta_2, s, \varepsilon, \delta)| \lesssim \frac{1}{|\lambda_2|}.$$

Therefore we argue in a similar way as in the proof of Lemma 3.3.4. First, we apply the method of stationary phase in x_1 and then use the fact that $\partial_2^m g(x^0) \neq 0$, in order to obtain the desired result

$$|I(\lambda_2, \beta_1, \beta_2, s, \varepsilon, \delta)| \lesssim \frac{1}{|s|^{\frac{1}{m}} |\lambda_2|^{\frac{1}{2}+\frac{1}{m}}}$$

if $0 < |s|$ is sufficiently small, β_1 is close to β_1^0, β_2 is small and $\chi(\cdot, \varepsilon, \sigma, \delta)$ is supported in a sufficiently small neighborhood of x^0. $\square$

At the end of this chapter we state the last lemma whose proof in large parts makes use of a deep result from [21] concerning estimates of oscillatory integral of degenerate Airy type.

This situation arises in the case if we try to establish a global estimate for Λ if $|\sigma| \sim |\varepsilon|$ and if in addition $\partial_2^2 g(x^0) = 0 \neq \partial_2 g(x^0)$.

Lemma 3.3.12. *Assume that*

$$\min\left\{\left|\partial_2 g(x^0)\right|, \left|\partial_1\partial_2 g(x^0)\right|\right\} > 0, \tag{3.21}$$

and that there is some $m \geq 3$ such that

$$\partial_2^l g(x^0) = 0 \quad \text{for } l = 2, \dots, m-1 \quad \text{and} \quad \partial_2^m g(x^0) \neq 0. \tag{3.22}$$

Then there exist a neighborhood $U(x^0)$ of x^0, a constant $b > 0$ and a neighborhood $\tilde{I}_1 \times \tilde{I}_2 \times \tilde{V} \subseteq I_1 \times I_2 \times V$ of the origin such that for every smooth function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\varepsilon|(1+|\lambda|)^{\frac{1}{2}+b}} \tag{3.23}$$

holds true for all $(\lambda, \varepsilon, \sigma, \delta) \in \mathbb{R}^3 \times \tilde{I}_1 \setminus \{0\} \times \tilde{I}_2 \setminus \{0\} \times \tilde{V}$ with $|\sigma| = |\varepsilon|$.

Proof. Assume that $|\varepsilon| = |\sigma|$. Since $\partial_2 g(x^0) \neq 0$, we see again that applying integration by parts in x_2 we get the estimate

$$|\Lambda(\lambda, \varepsilon, \sigma, \delta)| \lesssim \frac{1}{|\varepsilon|(1+|\lambda|)} \tag{3.24}$$

if either $|\lambda_2| \gg |\lambda_3|$ or $|\lambda_2| \ll |\lambda_3|$. Therefore we can assume that $|\lambda_2| \sim |\lambda_3|$. In such a case $|\lambda_2| \sim |\lambda|$. We rewrite Λ as

$$I(\lambda_2, \beta_1, \beta_2, \varepsilon, \delta) = \int\limits_{\mathbb{R}^2} e^{i\lambda_2(f_1(x_1,\beta_1,\beta_2,\delta)+\varepsilon f_2(x_1,x_2,\beta_2,\delta))}\chi(x, \varepsilon, \sigma, \delta)dx,$$

where

$$\beta_1 = \frac{\lambda_1}{\lambda_2}, \quad \beta_2 = \frac{\lambda_3}{\lambda_2},$$

$$f_1(x_1, \beta_1, \beta_2, \delta) = \beta_1 x_1 + c_1 x_1^{\gamma_1} + \beta_2 c_2 x_1^{\gamma_2} + R_1(x_1, \delta) + \beta_2 R_3(x_1, \delta),$$

$$f_2(x_1, x_2, \beta_2, \delta) = x_2 + \beta_2 g(x_1, x_2) + \beta_2 R_2(x_1, x_2, \delta).$$

Observe that $|\beta_1| \lesssim 1$. We can assume that β_2 lies in some small interval $(-\rho + \beta_2^0, \beta_2^0 + \rho)$,

$\beta_2^0 = -\frac{1}{\partial_2 g(x^0)}$, $\rho \ll 1$, since otherwise we just apply integration by parts in x_2 and conclude (3.24). For

$$\delta_{k+1} = \beta_2 - \beta_2^0 \in (-\rho, \rho)$$

we write

$$\begin{aligned} f_2(x_1, x_2, \beta_2, \delta) &= \tilde{f}_2(x_1, x_2, \delta, \delta_{k+1}) \\ &= x_2 + \beta_2^0 g(x) + \underbrace{\delta_{k+1} g(x) + \delta_{k+1} R_2(x_1, x_2, \delta) + \beta_2^0 R_2(x_1, x_2, \delta)}_{R(x_1, x_2, \delta, \delta_{k+1})}. \end{aligned} \tag{3.25}$$

The function R is obviously a smooth perturbation in (δ, δ_{k+1}). Now, we see that because of (3.21) and (3.22) we get

$$\partial_2^l \tilde{f}_2(x^0, 0) = 0, \quad l = 1, \dots, m-1, \quad \partial_1 \partial_2 \tilde{f}_2(x^0, 0) \neq 0 \neq \partial_2^m \tilde{f}_2(x^0, 0).$$

By Proposition 3.3.5 we see that if we set

$$\tilde{f}_1(x_1, \beta_1, \delta, \delta_{k+1}) = \beta_1 x_1 + c_1 x_1^{\gamma_1} + \beta_2^0 c_2 x_1^{\gamma_2} + \delta_{k+1} c_2 x_1^{\gamma_2} + R_1(x_1, \delta) + \beta_2^0 R_3(x_1, \delta) + \delta_{k+1} R_3(x_1, \delta),$$

then

$$\left|\partial_1^2 \tilde{f}_1(x_1^0, \beta_1, 0, 0)\right| + \left|\partial_1^3 \tilde{f}_1(x_1^0, \beta_1, 0, 0)\right| > 0$$

holds true uniformly in β_1. Thus we can apply Theorem 5.4 in [21] and conclude that there is a small neighborhood $U(x^0)$ of x^0, small numbers α, ρ, $b > 0$ and a small neighborhood $\tilde{V} \subseteq V$ of the origin such that for any function $\chi(\cdot, \varepsilon, \sigma, \delta)$ supported in $U(x^0)$ the estimate

$$|I(\lambda_2, \beta_1, \beta_2, \varepsilon, \delta)| \lesssim \frac{1}{|\lambda_2|^{\frac{1}{2}+b} |\varepsilon|^{l_m + c_m b}}$$

holds true for $|\sigma| = |\varepsilon|$, σ, $\varepsilon \in (-\alpha, \alpha)^2$, $\beta_2 \in (\beta_2^0 - \rho, \beta_2^0 + \rho)$ and $\delta \in \tilde{V}$. Here is

$$l_k = \begin{cases} \frac{1}{6}, & \text{for } k \leq 5, \\ \frac{k-3}{2(2k-3)}, & \text{for } k \geq 6, \end{cases} \quad \text{and} \quad c_k = \begin{cases} 1, & \text{for } k \leq 5, \\ 2, & \text{for } k \geq 6. \end{cases}$$

In [21] the above estimate was proved for $x^0 = 0$ but this is achieved by shifting coordinates. Since $|\lambda_2| \sim |\lambda|$ and $l_m \in [\frac{1}{6}, \frac{1}{4}]$, we immediately conclude (3.23) if $b > 0$ is small enough. □

4 Estimate for the maximal average over an analytic surface in $\mathbb{R}^3$

In this chapter we shall present the first part of the proof of Theorem 1.3.1. The next theorem, which goes back to E. M. Stein, will play a fundamental role in the proof.

Theorem 4.0.13. *Let τ be a tempered distribution supported in a ball $B \subseteq \mathbb{R}^3$ of radius C_1 and assume that $\widehat{\tau}$ is a smooth function. Furthermore, assume that $|\widehat{\tau}(\lambda)| \leq C_1$ and $\max\{|x| : x \in \operatorname{supp} \tau\} \leq C_2$. Let D_t^a be the family of the non-isotropic dilations defined in (1.3). Suppose that*

$$\left(\int_{\frac{1}{2}}^{2} |\widehat{\tau}(D_t^a(\lambda))|^2 \, dt \right)^{\frac{1}{2}} \leq C_1 (1 + |\lambda|)^{-\frac{1}{2}} \gamma(|\lambda|), \tag{4.1}$$

$$\left(\int_{\frac{1}{2}}^{2} |\nabla \widehat{\tau}(D_t^a(\lambda))|^2 \, dt \right)^{\frac{1}{2}} \leq C_2 (1 + |\lambda|)^{-\frac{1}{2}} \gamma(|\lambda|), \tag{4.2}$$

where γ is bounded and decreasing on $[0, \infty)$, and in addition satisfies $C_\gamma = \sum\limits_{k=0}^{\infty} \gamma(2^k) < \infty$. Define $\widehat{\tau}_t(\lambda) = \widehat{\tau}(D_t^a(\lambda))$ and set

$$\mathcal{M}_\tau f(x) = \sup_{t>0} |f * \tau_t(x)|.$$

Then there exists a positive constant $C = C(C_\gamma, a)$ such that

$$\|\mathcal{M}_\tau f\|_{L^2(\mathbb{R}^3)} \leq C \sqrt{C_1 C_2} \|f\|_{L^2(\mathbb{R}^3)}$$

holds true for any $f \in \mathcal{S}$.

Remark 4.0.14. *Observe that the estimates* (4.1) *and* (4.2) *follow from the uniform estimate*

$$|\widehat{\tau}(\lambda)| + |\nabla\widehat{\tau}(\lambda)| \leq \frac{C}{(1+|\lambda|)^{\frac{1}{2}+\delta}}, \quad C, \delta > 0, \ \lambda \in \mathbb{R}^3. \tag{4.3}$$

In the end we shall establish uniform estimates of the form (4.3) *instead of the averaged estimates* (4.1), (4.2).

For the proof of Theorem 4.0.13 we refer the reader to [26].

4.1 Outline of the proof

The proof of Theorem 1.3.1 consists of an algorithm of the resolution of singularities of the function $\partial_2\varphi$, where φ is an analytic function as in Theorem 1.3.1. In this chapter we shall give a precise description of each step of this algorithm. After finitely many steps the algorithm always leads to the estimate of certain maximal operators which can be classified in different categories. After appropriate dyadic decompositions the desired L^p-boundedness of the maximal operator follows from the uniform estimate of the Fourier transform of the measure, cf. Theorem 4.0.13. This leads to the estimates of the oscillatory integrals, where due to the dyadic decomposition the phase function contains additional smooth perturbations. These estimates can be found in Chapter 3.
There will be one degenerate situation, where the described procedure requires other arguments. This situation is very similar to the stopping time algorithm in Chapter 9 of the article [21]. We will adapt these ideas from the stopping time algorithm to the underlying situation.
First, we observe that since the pointwise estimate

$$\mathcal{M}f \leq \mathcal{M}|f|$$

holds true, we can and shall always assume that the function f is non-negative. Since the case $\varphi \equiv 0$ is trivial, we can assume that $\mathcal{T}(\varphi) \neq \emptyset$. Basically, we will prove that there is a finite set $\mathcal{I}$ depending on φ and a family of maximal operators $\mathcal{M}_i$, $i \in \mathcal{I}$, each of which is bounded on $L^p(\mathbb{R}^3)$ for every $p > 2$, and the pointwise estimate

$$\mathcal{M}f \leq \sum_{i\in\mathcal{I}} \mathcal{M}_i f$$

holds true for every bounded positive function f. In order to start the algorithm of the resolution of singularities, we first describe a preparation step, which is different from further steps.

4.2 Preparation step

Changing variables

$$(x_1, x_2) \longmapsto (-x_1, x_2)$$

we reduce the problem to the right half-plane

$$\mathcal{H} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \; x_1 > 0 \right\}.$$

Thus we need to obtain the desired L^p-estimate for the maximal operator $\mathcal{M}$ given by

$$\mathcal{M}f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x))\psi(x)dx, \quad f \geq 0, \tag{4.4}$$

where $\psi \in C_0^\infty(\Omega)$ is any smooth non-negative function. Without loss of generality we can assume that ψ is identically one on some small neighborhood of the origin lying in Ω. During the proof we will put some restrictions on the size of Ω, in order to make sure that certain dyadic parameters satisfies certain size conditions. By assumption, the function φ is analytic and we write

$$\varphi(x_1, x_2) = \sum_{\alpha,\beta=0}^{\infty} c_{\alpha,\beta} x_1^\alpha x_2^\beta, \quad (x_1, x_2) \in \Omega.$$

Recall that

$$\varphi(0,0) = 0, \quad \nabla\varphi(0,0) = (0,0).$$

The arguments in the proof will be based on the geometry of the Newton polyhedron. Assume first that

$$\mathcal{N}(\varphi) = \{(t_1, t_2) : \; t_1 \geq A, \; t_2 \geq B\},$$

i.e. $\mathcal{N}_d(\varphi) = \{(A, B)\}$. Since φ is an analytic function, we conclude that

$$\varphi(x_1, x_2) = c_{A,B} x_1^A x_2^B + R(x_1, x_2), \quad c_{A,B} \in \mathbb{R} \setminus \{0\},$$

and

$$R(x_1, x_2) = \sum_{\substack{(\alpha,\beta)\in\mathcal{T}(\varphi)\\(\alpha,\beta)\neq(A,B)}} c_{\alpha,\beta} x_1^\alpha x_2^\beta$$

is the analytic remainder term. In this case we decompose bi-dyadically, i.e. dyadically in each variable. For this purpose consider a dyadic partition of unity

$$\sum_{k=N}^{\infty} \chi_k(s) = 1, \quad \text{for } s \in [-2^{-N}, 0) \cup (0, 2^{-N}].$$

The function χ is a positive smooth function supported in $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and we set $\chi_k = \chi(2^k \cdot)$. Observe that if Ω is assumed to be sufficiently small, then the number $N \in \mathbb{N}$ can be assumed to be sufficiently large satisfying

$$\pi_1(\Omega \cap \mathcal{H}) \subseteq [0, 2^{-N}], \quad \pi_2(\Omega \cap \mathcal{H}) \subseteq [-2^{-N}, 2^{-N}].$$

Then we can estimate the maximal operator $\mathcal{M}$ in (4.4) by

$$\begin{aligned}
\mathcal{M}f(\cdot) &= \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x)) \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} \chi_j(x_1)\chi_k(x_2)\psi(x)dx \\
&\leq \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x))\chi_j(x_1)\chi_k(x_2)\psi(x)dx \\
&= \sum_{j,k=N}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(2^{-j}x_1, 2^{-k}x_2))\chi(x_1)\chi(x_2)\psi(2^{-j}x_1, 2^{-k}x_2)dx \\
&= \sum_{j,k=N}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - tA_{j,k}\Phi_{j,k}(x))\chi \otimes \chi(x)\psi(2^{-j}x_1, 2^{-k}x_2)dx,
\end{aligned}$$

where

$$\begin{aligned}
A_{j,k}(z_1, z_2, z_3) &= (2^{-j}z_1, 2^{-k}z_2, 2^{-jA-kB}z_3), \\
\chi \otimes \chi(x_1, x_2) &= \chi(x_1)\chi(x_2), \qquad (4.5)
\end{aligned}$$

and

$$\Phi_{j,k}(x_1, x_2) = \left(x_1, x_2, c_{A,B}x_1^A x_2^B + 2^{jA+kB}R(2^{-j}x_1, 2^{-k}x_2)\right).$$

Clearly,

$$\operatorname{supp} \chi \otimes \chi \subseteq \left\{x_1 : \; 2^{-1} \leq |x_1| \leq 2\right\} \times \left\{x_2 : \; 2^{-1} \leq |x_2| \leq 2\right\}. \tag{4.6}$$

Observe that

$$2^{jA+kB} R(2^{-j}x_1, 2^{-k}x_2) = \mathcal{O}(\max\{2^{-j}, 2^{-k}\}) \tag{4.7}$$

in C^K for every $K \in \mathbb{N}_0$. To be very precise, notice that we have the identity

$$2^{jA+kB} R(2^{-j}x_1, 2^{-k}x_2) = \tilde{R}(x_1, x_2, 2^{-j}, 2^{-k}),$$

where

$$\tilde{R}(x_1, x_2, \delta_1, \delta_2) = x_1^A x_2^B \sum_{\alpha+\beta \geq 1} c_{A+\alpha, B+\beta} (\delta_1 x_1)^\alpha (\delta_2 x_2)^\beta.$$

The function $\tilde{R}$ is smooth on $(-4,4)^2 \times (-\epsilon, \epsilon)^2$ for some small ϵ and every derivative in x satisfies

$$D_x^\gamma \tilde{R}(x_1, x_2, \delta_1, \delta_2) = \mathcal{O}(\max\{|\delta_1|, |\delta_2|\}).$$

In the sequel we shall refer to such functions as perturbation terms or only use the abbreviated notation as in (4.7) with Landau's symbol.

Remark 4.2.1. *Observe that the above arguments fail to be true if the function φ is only assumed to be of finite type. Consider*

$$\varphi(x_1, x_2) = \eta(x_1) + x_2^m,$$

where η is some some smooth flat function, e.g. $\eta(x_1) = \mathrm{e}^{-\frac{1}{x_1}} \mathbf{1}_{(0,\infty)}(x_1)$. Then

$$\mathcal{N}_d(\varphi) = \{(t_1, t_2) : \; t_2 \geq m\}.$$

If we apply the bi-dyadic decomposition and rescale as before, we see that the coordinates are given by

$$\left(x_1, x_2, 2^{km}\eta(2^{-j}x_1) + x_2^m\right).$$

The term $2^{km}\eta(2^{-j}x_1)$ can be very large for $x_1 \sim 1$ and we cannot apply previous arguments. In fact, in Section 6.2 we shall prove that the maximal average is not bounded on $L^p(\mathbb{R}^3)$ for $p > 2$, if the surface is only assumed to be of finite type.

In view of Lemma 2.2.1 the desired L^p-boundedness of the maximal operator $\mathcal{M}$ in (4.4) follows from the L^p-boundedness of

$$\mathcal{M}^{j,k}f(\cdot) = 2^{-j-k}\sup_{t>0}\int_{\mathcal{H}} f(\cdot - t\Phi_{j,k}(x))\chi\otimes\chi(x)\psi(2^{-j}x_1, 2^{-k}x_2)dx,$$

if the L^p-norm of each $\mathcal{M}^{j,k}$ is at most a constant multiple of 2^{-j-k}. Since $A+B\geq 2$, the Hessian of the monomial $c_{A,B}x_1^A x_2^B$ does not vanish on the support of $\chi\otimes\chi$. The same holds true for

$$c_{A,B}x_1^A x_2^B + 2^{jA+kB}R(2^{-j}x_1, 2^{-k}x_2),$$

uniformly in (j,k), or more precisely, there exists a positive constant C, a positive integer N and $\gamma=(\gamma_1,\gamma_2)\in\mathbb{N}_0^2$ with $\gamma_1+\gamma_2=2$ such that

$$\inf_{x\in\operatorname{supp}\chi\otimes\chi}\inf_{j\geq N}\inf_{k\geq N}\left|D_x^\gamma\left(c_{A,B}x_1^A x_2^B + 2^{jA+kB}R(2^{-j}x_1, 2^{-k}x_2)\right)\right|\geq C.$$

This means that the corresponding hypersurface has at least one non-vanishing principle curvature at every point. The desired L^p-boundedness follows from the article of C. D. Sogge [40].

Next, we shall assume that the Newton diagram $\mathcal{N}_d(\varphi)$ has at least one edge. Denote the vertices of $\mathcal{N}_d(\varphi)$ by (A_l, B_l), $l=0,\dots,n$, $n\geq 1$. We assume $(A_i)_i$ to be strictly increasing. For the line L_l passing through the points (A_{l-1}, B_{l-1}) and (A_l, B_l), $l=1,\dots,n$, there exists a unique weight

$$\tilde{\kappa}^{(l)} = (\tilde{\kappa}_1^{(l)}, \tilde{\kappa}_2^{(l)}) = \left(\frac{q_l}{m_l}, \frac{p_l}{m_l}\right), \quad \gcd(q_l, p_l) = 1,$$

such that

$$L_l = \left\{(t_1,t_2):\ \tilde{\kappa}_1^{(l)}t_1 + \tilde{\kappa}_2^{(l)}t_2 = 1\right\}.$$

Then the absolute value of the slope of the line L_l is then obviously given by $\frac{\tilde{\kappa}_1^{(l)}}{\tilde{\kappa}_2^{(l)}}$. Let

$$s_l = \frac{\tilde{\kappa}_2^{(l)}}{\tilde{\kappa}_1^{(l)}} = \frac{p_l}{q_l}$$

be the absolute value of the reciprocal of the slope of the line L_l. Observe that by the geometry of $\mathcal{N}_d(\varphi)$ we conclude that $(s_l)_l$ is strictly increasing. Let $M\in\mathbb{N}$ be a large number. We decompose the set $\Omega\cap\mathcal{H}$ in subsets T_l which are transition domains between two different homogeneities and domains H_l which correspond to the homogeneous part of φ with respect to the weight $\tilde{\kappa}^{(l)}$. The transition domains T_l are given by

$$T_0 = \left\{(x_1, x_2) \in \mathcal{H} \cap \Omega : \; 2^M x_1^{s_1} < |x_2|\right\},$$

$$T_l = \left\{(x_1, x_2) \in \mathcal{H} \cap \Omega : \; 2^M x_1^{s_{l+1}} < |x_2| \leq 2^{-M} x_1^{s_l}\right\}, \quad l = 1, \dots, n-1,$$

$$T_n = \left\{(x_1, x_2) \in \mathcal{H} \cap \Omega : \; |x_2| \leq 2^{-M} x_1^{s_n}\right\}.$$

The homogeneous domains H_l are defined by

$$H_l = \left\{(x_1, x_2) \in \mathcal{H} \cap \Omega : \; 2^{-M} x_1^{s_l} < |x_2| \leq 2^M x_1^{s_l}\right\}, \quad l = 1, \dots, n.$$

Clearly, the domains T_0 and T_n are also homogeneous, but we can interpret them as transition domains if we formally set $s_0 = 0$ and $s_{n+1} = \infty$. It is evident that

$$\mathcal{H} \cap \Omega = \bigcup_{l=0}^{n} T_l \cup \bigcup_{l=1}^{n} H_l.$$

Observe that if Ω is chosen small enough, then the number M can be chosen sufficiently large and the above domains are disjoint. A similar decomposition of the domain goes back to D. H. Phong and E. M. Stein [36], where the authors consider oscillatory integral operators.

In order to localize the coordinates to these domains, let

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } [-1, 1], \quad \operatorname{supp} \eta \subseteq [-2, 2]. \tag{4.8}$$

For $x \in \mathcal{H}$ let

$$h_l(x_1, x_2) = \eta\left(\frac{x_2}{2^M x_1^{s_l}}\right)\left(1 - \eta\left(\frac{2x_2}{2^{-M} x_1^{s_l}}\right)\right), \quad l = 1, \dots, n.$$

In order to localize to the domains $(T_l)_l$, we set

$$\tau_l(x_1, x_2) = \eta\left(\frac{x_2}{2^{-M} x_1^{s_l}}\right)\left(1 - \eta\left(\frac{2x_2}{2^M x_1^{s_{l+1}}}\right)\right), \quad l = 1, \dots, n-1,$$

$$\tau_0(x_1, x_2) = 1 - \eta\left(\frac{2x_2}{2^M x_1^{s_1}}\right),$$

$$\tau_n(x_1, x_2) = \eta\left(\frac{x_2}{2^{-M}x_1^{s_n}}\right).$$

Then obviously

$$h_l\Big|_{H_l} \equiv 1 \equiv \tau_l\Big|_{T_l}, \quad 0 \leq h_l \leq 1, \quad 0 \leq \tau_l \leq 1.$$

We get

$$\mathcal{M}f \leq \sum_{l=1}^{n} \mathcal{M}_{h_l} f + \sum_{l=0}^{n} \mathcal{M}_{\tau_l} f,$$

where

$$\mathcal{M}_{h_l} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x))\psi(x)h_l(x)dx,$$

$$\mathcal{M}_{\tau_l} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x))\psi(x)\tau_l(x)dx.$$

We shall first describe the argument for the transition domains $(T_l)_l$. The proof is again based on the bi-dyadic decomposition. Arguing as in the previous case we get for every $l \in \{0, \ldots, n\}$

$$\mathcal{M}_{\tau_l} f(\cdot) \leq \sum_{j,k=N}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(2^{-j}x_1, 2^{-k}x_2))\chi \otimes \chi(x)(\psi\tau_l)(2^{-j}x_1, 2^{-k}x_2)dx,$$

where the positive integer N is large.

Lemma 4.2.2. *For every j, $k \geq N$ with $\tau_l(2^{-j}\cdot, 2^{-k}\cdot)\chi \otimes \chi \neq 0$ we have*

$$\varphi(2^{-j}x_1, 2^{-k}x_2) = 2^{-jA_l - kB_l}(c_{A_l,B_l}x_1^{A_l}x_2^{B_l} + R^l_{j,k}(x_1, x_2)),$$

where $R^l_{j,k}$ is a smooth perturbation term.

Proof. First, observe that for every $l \in \{1, \ldots, n\}$ and for some $x \in \mathcal{H}$ with

$$\chi \otimes \chi(x)\tau_l(2^{-j}x_1, 2^{-k}x_2) \neq 0,$$

we conclude using (4.6) and (4.8)

$$2^{-k-1} \leq 2^{-k}|x_2| \leq 2 \cdot 2^{-M}2^{-js_l}x_1^{s_l} \leq 2^{1-M}2^{-js_l}2^{s_l}.$$

For $M \geq \max\{2(s_l+5):\ l=1,\dots,n\}$ we conclude

$$2^{-k+js_l} \leq 2^{2-M+s_l} \leq 2^{-\frac{M}{2}}. \tag{4.9}$$

For each $l \in \{0,\dots,n-1\}$ we conclude in a similar manner

$$2^2 \cdot 2^{-k} \geq 2 \cdot 2^{-k}|x_2| \geq 2^{-js_{l+1}} 2^M x_1^{s_{l+1}} \geq 2^{-js_{l+1}} 2^M 2^{-s_{l+1}},$$

and therefore

$$2^{k-js_{l+1}} \leq 2^{2-M+s_{l+1}} \leq 2^{-\frac{M}{2}}. \tag{4.10}$$

Now assume that $l \in \{1,\dots,n-1\}$. The case $l \in \{0,n\}$ is similar. We have already seen that the remainder term corresponding to the quadrant

$$\{(t_1,t_2):\ t_1 \geq A_l,\, t_2 \geq B_l,\, t_1+t_2 > A_l+B_l\}$$

is a smooth perturbation in $(2^{-j},2^{-k})$. We show that the remainder term corresponding to the set

$$S_l = \{(t_1,t_2):\ t_1 < A_l,\, t_2 > B_l\}$$

is a smooth perturbation term in $(2^{-\frac{j}{q_l}}, \sigma_{j,k,l})$, where $\sigma_{j,k,l} = 2^{-k+js_l} \leq 2^{-\frac{M}{2}}$. The arguments for the set $\{(t_1,t_2):\ t_1 > A_l,\, t_2 < B_l\}$ are similar. By the geometry of the Newton polyhedron we have

$$\mathcal{T}(\varphi) \cap \{\{(t_1,t_2):\ t_1 < A_l,\, t_2 > B_l\} \subseteq \bigcup_{r=0}^{\infty} L_{m_l+r},$$

where

$$L_{m_l+r} = \{(t_1,t_2) \in \mathbb{N}_0^2:\ q_l t_1 + p_l t_2 = m_l + r,\, t_1 < A_l,\, t_2 > B_l\}.$$

Let

$$(x_1,x_2) \in \left(4^{-1},4\right) \times \left\{y:\ 4^{-1} < |y| < 4\right\} \supset \operatorname{supp} \chi \otimes \chi \cap \mathcal{H}.$$

Rewrite the part of the term $R^l_{j,k}(x_1,x_2)$ corresponding to the set S_l as

$$2^{jA_l+kB_l} \sum_{(\alpha,\beta)\in S_l} c_{\alpha,\beta} 2^{-j\alpha-k\beta} x_1^\alpha x_2^\beta = \sum_{r=0}^{\infty} \sum_{(\alpha,\beta)\in L_{m_l+r}} 2^{jA_l+kB_l-j\alpha-k\beta} c_{\alpha,\beta} x_1^\alpha x_2^\beta.$$

Let $S_r^{(l)} = \pi_2(L_{m_l+r})$. For every $r \geq 0$ and every $\beta \in S_r^{(l)}$ there is a unique $\alpha = \alpha(r,\beta) \in \mathbb{N}_0$ such that $(\alpha,\beta) \in L_{m_l+r}$. In particular, $q_l\alpha = m_l + r - p_l\beta$. We write $d_{r,\beta} = c_{\alpha,\beta}$. This gives

$$\begin{aligned} 2^{jA_l+kB_l} \sum_{(\alpha,\beta)\in S_l} c_{\alpha,\beta} 2^{-j\alpha-k\beta} x_1^\alpha x_2^\beta &= \sum_{r=0}^\infty \sum_{(\alpha,\beta)\in L_{m_l+r}} 2^{jA_l+kB_l-j\alpha-k\beta} c_{\alpha,\beta} x_1^\alpha x_2^\beta \\ &= \sum_{r=0}^\infty \sum_{\beta\in S_r^{(l)}} 2^{jA_l+kB_l-k\beta} (2^{-\frac{j}{q_l}})^{m_l+r-p_l\beta} d_{r,\beta} x_1^{\frac{m_l+r-p_l\beta}{q_l}} x_2^\beta. \end{aligned}$$

Using $A_l q_l + B_l p_l = m_l$, we get

$$\begin{aligned} & jA_l + kB_l - k\beta - \tfrac{j}{q_l}(m_l + r - p_l\beta) \\ = \;& jA_l + kB_l - k\beta - jA_l - j\tfrac{p_l}{q_l}B_l - j\tfrac{r}{q_l} + j\tfrac{p_l}{q_l}\beta \\ = \;& k(B_l - \beta) + js_l(\beta - B_l) - j\tfrac{r}{q_l} \\ = \;& (-k + js_l)(\beta - B_l) - j\tfrac{r}{q_l}. \end{aligned}$$

Eventually, this gives

$$\begin{aligned} & 2^{jA_l+kB_l} \sum_{(\alpha,\beta)\in S_l} c_{\alpha,\beta} 2^{-j\alpha-k\beta} x_1^\alpha x_2^\beta \\ = \;& \sum_{r=0}^\infty \sum_{\beta\in S_r^{(l)}} d_{r,\beta} 2^{jA_l+kB_l-k\beta} (2^{-\frac{j}{q_l}})^{m_l+r-p_l\beta} x_1^{\frac{m_l+r-p_l\beta}{q_l}} x_2^\beta \\ = \;& \sum_{r=0}^\infty \sum_{\beta\in S_r^{(l)}} d_{r,\beta} (2^{-\frac{j}{q_l}})^r (\sigma_{j,k,l})^{\beta-B_l} x_1^{\frac{m_l+r-p_l\beta}{q_l}} x_2^\beta \\ = \;& x_1^{\frac{m_l}{q_l}} \left(\frac{x_2}{x_1^{\frac{p_l}{q_l}}}\right)^{B_l} \sum_{r=0}^\infty (2^{-\frac{j}{q_l}})^r x_1^{\frac{r}{q_l}} \sum_{\beta\in S_r^{(l)}} d_{r,\beta} (\sigma_{j,k,l})^{\beta-B_l} \left(\frac{x_2}{x_1^{\frac{p_l}{q_l}}}\right)^{\beta-B_l} \\ = \;& x_1^{\frac{m_l}{q_l}} \left(\frac{x_2}{x_1^{\frac{p_l}{q_l}}}\right)^{B_l} \sum_{r=0}^\infty G_r\left(2^{-\frac{j}{q_l}} x_1^{\frac{1}{q_l}}\right) F_r\left(\sigma_{j,k,l} \cdot \frac{x_2}{x_1^{\frac{p_l}{q_l}}}\right), \end{aligned}$$

where $G_r(y) = y^r$ and $F_r(y) = \sum_{\beta\in S_r^{(l)}} d_{r,\beta} y^{\beta-B_l}$. This gives the desired result, since each $\beta \in S_r^{(l)}$ satisfies $\beta > B_l$. $\square$

We have

$$\begin{aligned}\mathcal{M}_{\tau_l} f(\cdot) &\leq \sum_{j,k=N}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(2^{-j}x_1, 2^{-k}x_2))\chi \otimes \chi(x)(\psi\tau_l)(2^{-j}x_1, 2^{-k}x_2)dx \\ &= \sum_{j,k=N}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - tA^l_{j,k}\Phi^l_{j,k}(x_1, x_2))\chi \otimes \chi(x)(\psi\tau_l)(2^{-j}x_1, 2^{-k}x_2)dx,\end{aligned}$$

where

$$A^l_{j,k}(z_1, z_2, z_3) = (2^{-j}z_1, 2^{-k}z_2, 2^{-jA_l-kB_l}z_3),$$

$$\Phi^l_{j,k}(x_1, x_2) = \left(x_1, x_2, c_{A_l,B_l}x_1^{A_l}x_2^{B_l} + R^l_{j,k}(x_1, x_2)\right).$$

Recall that by assumption $\nabla\varphi(0,0) = (0,0)$. Therefore $A_l + B_l \geq 2$, and therefore the Hessian of the monomial $c_{A_l,B_l}x_1^{A_l}x_2^{B_l}$ does not vanish on $\operatorname{supp}\chi \otimes \chi$. As in the first case we conclude the desired result from Lemma 2.2.1 and [40].

Next, we turn our attention to the homogeneous domains H_l, $l = 1, \dots, n$. In each of these domains we shall do further decompositions. This is done as follows. For $\tilde{\kappa}^{(l)} = (\tilde{\kappa}_1^{(l)}, \tilde{\kappa}_2^{(l)})$ define the homogeneous dilation $\delta^{\tilde{\kappa}^{(l)}}$ on $\mathbb{R}^2$ by

$$\delta_r^{\tilde{\kappa}^{(l)}}(x_1, x_2) = (r^{\tilde{\kappa}_1^{(l)}}x_1, r^{\tilde{\kappa}_2^{(l)}}x_2), \quad r > 0.$$

Denote by $\varphi_{\tilde{\kappa}^{(l)}}$ the $\tilde{\kappa}^{(l)}$-homogeneous part of degree one of φ, i.e.

$$\varphi_{\tilde{\kappa}^{(l)}}(x_1, x_2) = \sum_{(\alpha,\beta)\in L_l\cap\mathbb{N}_0^2} c_{\alpha,\beta}x_1^{\alpha}x_2^{\beta}.$$

Then clearly, $\varphi\left(\delta_r^{\tilde{\kappa}^{(l)}}(x)\right) = r\varphi_{\tilde{\kappa}^{(l)}}(x)$ for every $x \in \mathbb{R}^2$, $r > 0$. For each $l \in \{1, \dots, n\}$ we decompose φ as in (2.2)

$$\varphi = \varphi_{\tilde{\kappa}^{(l)}} + R_{\tilde{\kappa}^{(l)}},$$

where $R_{\tilde{\kappa}^{(l)}} = \varphi - \varphi_{\tilde{\kappa}^{(l)}}$ is the analytic remainder term of φ consisting only of terms of higher $\tilde{\kappa}^{(l)}$-degree, i.e. $\mathcal{T}(R_{\tilde{\kappa}^{(l)}})$ contains only those points $(\alpha, \beta) \in \mathcal{T}(\varphi)$ lying strictly above the line L_l.

The second part of the preparation step consists in further refined decompositions of the domain H_l. Recall that since each L_l contains at least two points of $\mathcal{T}(\varphi)$, each polynomial

$\varphi_{\tilde{\kappa}^{(l)}}$ is not a monomial. By Lemma 2.2.5 we write

$$\varphi_{\tilde{\kappa}^{(l)}}(x_1, x_2) = C_l x_1^{\alpha_l} x_2^{\beta_l} \prod_{j=1}^{M_l} (x_2^{q_l} - \lambda_j^{(l)} x_1^{p_l})^{n_j^{(l)}}. \tag{4.11}$$

The points (A_l, B_l) and the numbers α_l, β_l, $n_j^{(l)}$ have also a certain algebraic connection which we shall not specify here.
Next, we will identify "bad" and "good" points in each domain H_l. The "bad" points are namely those points in $\mathcal{H}$ where the gradient of $\varphi_{\tilde{\kappa}^{(l)}}$ vanishes. Assume there exists a point $x^0 = (x_1^0, x_2^0) \in \mathcal{H}$ with $\nabla\varphi_{\tilde{\kappa}_l}(x^0) = (0,0)$. By Euler's homogeneity relation, see Lemma 2.2.4, we see that $\varphi_{\tilde{\kappa}^{(l)}}(x^0) = 0$. Every zero of $\varphi_{\tilde{\kappa}^{(l)}}$ in $\mathcal{H}$ which does not lie on a coordinate axis lies on some curve

$$\mathcal{C}_j^{(l)} = \left\{ (r, (\lambda_j^{(l)})^{\frac{1}{q_l}} r^{s_l}) : \ r > 0 \right\}, \quad \lambda_j^{(l)} \in \mathbb{R} \setminus \{0\}.$$

If for some $x^0 \in \operatorname{supp} h_l \cap \mathcal{H}$

$$\nabla\varphi_{\tilde{\kappa}_l}(x^0) = (0,0),$$

then we conclude that $x^0 = (r_0, (\lambda_j^{(l)})^{\frac{1}{q_l}} r_0^{s_l})$ for some $r_0 > 0$, and the corresponding multiplicity $n_j^{(l)}$ is at least two. Set $z_{l,j} = (\lambda_j^{(l)})^{\frac{1}{q_l}}$. In order to localize the coordinates to the curves $\mathcal{C}_j^{(l)}$ with the corresponding multiplicity $n_j^{(l)} \geq 2$, let

$$\eta_{l,j}(x_1, x_2) = \eta \left(\frac{x_2 - z_{l,j} x_1^{s_l}}{\varepsilon_l x_1^{s_l}} \right),$$

with η defined in (4.8) and $\varepsilon_l > 0$ is sufficiently small. For each l we obtain

$$\mathcal{M}_{h_l} f \leq \sum_j \mathcal{M}_{\eta_{l,j}} f + \overline{\mathcal{M}}_l f,$$

where

$$\mathcal{M}_{\eta_{l,j}} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x)) \psi(x) \eta_{l,j}(x) dx,$$

$$\overline{\mathcal{M}}_l f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x)) (\psi h_l (1 - \sum_j \eta_{l,j}))(x) dx.$$

We used that $h_l \leq 1$. In order to estimate $\mathcal{M}_{\eta_{l,j}}$, we shall proceed to the next step changing variables. First, we give the argument for the maximal operator $\overline{\mathcal{M}}_l$. This is achieved by dyadic decomposition adapted to the homogeneous dilation $\delta^{\tilde{\kappa}^{(l)}}$. Using Lemma A.0.2 we write

$$\sum_{k=N}^{\infty} \rho_k(x) = 1 \quad \text{for } x \in \Omega \setminus \{0\},$$

where

$$\rho_k = \rho(\delta^{\tilde{\kappa}^{(l)}}_{2^k}(\cdot)),\ \rho \in C_0^\infty(\mathbb{R}^2),\ 0 \le \rho \le 1,\ \operatorname{supp} \rho \subseteq \left\{x \in \mathbb{R}^2 :\ 1 \le |x| \le 3\right\}.$$

The functions h_l, $\eta_{l,j}$ are $\tilde{\kappa}^{(l)}$-homogeneous of degree zero. Let

$$\overline{h}_l = h_l \left(1 - \sum_j \eta_{l,j}\right).$$

We obtain

$$\begin{aligned}
\overline{\mathcal{M}}_l f(\cdot) &= \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x))\psi(x) \sum_{k=N}^{\infty} \rho_k(x)\overline{h}_l(x)dx \\
&\le \sum_{k=N}^{\infty} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(x))\psi(x)\rho_k(x)\overline{h}_l(x)dx \\
&\le \sum_{k=N}^{\infty} 2^{-k(\tilde{\kappa}_1^{(l)}+\tilde{\kappa}_2^{(l)})} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - t\Phi(\delta^{\tilde{\kappa}^{(l)}}_{2^{-k}}(x)))\psi(\delta^{\tilde{\kappa}^{(l)}}_{2^{-k}}(x))\rho(x)\overline{h}_l(x)dx,
\end{aligned}$$

where in the last step we changed variables. Notice that assuming Ω to be a sufficiently small neighborhood of the origin, we can assume N and therefore k to be sufficiently large. We write

$$\Phi(\delta^{\tilde{\kappa}^{(l)}}_{2^{-k}}(x)) = A_{k,l}(\Phi_{k,l}(x)),$$

where

$$A_{k,l}(z_1, z_2, z_3) = (2^{-k\tilde{\kappa}_1^{(l)}} z_1, 2^{-k\tilde{\kappa}_2^{(l)}} z_2, 2^{-k} z_3),$$

$$\Phi_{k,l}(x_1, x_2) = \left(x_1, x_2, \varphi_{\tilde{\kappa}^{(l)}}(x_1, x_2) + 2^k R_{\tilde{\kappa}^{(l)}}(\delta^{\tilde{\kappa}^{(l)}}_{2^{-k}}(x))\right).$$

It is evident that $2^k R_{\tilde{\kappa}^{(l)}}(\delta^{\tilde{\kappa}^{(l)}}_{2^{-k}}(\cdot)) = \mathcal{O}(2^{-\delta_l k})$ for some $\delta_l > 0$. Using Lemma 2.2.1 we conclude that it is sufficient to prove that the L^p-norm, $p > 2$, of every maximal operator

$$\overline{\mathcal{M}}_{k,l} f(\cdot) = 2^{-k(\tilde{\kappa}_1^{(l)}+\tilde{\kappa}_2^{(l)})} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - t\Phi_{k,l}(x))\psi(\delta_{2^{-k}}^{\tilde{\kappa}^{(l)}}(x))\nu(x)\rho(x)\overline{h}_l(x)dx$$

is bounded by a constant multiple of $2^{-k(\tilde{\kappa}_1^{(l)}+\tilde{\kappa}_2^{(l)})}$. The function ν is a suitable bump function supported in $\mathcal{H}$ and is identically one on $\operatorname{supp} \rho\overline{h}_l \cap \mathcal{H}$. We remark that $\operatorname{supp} \nu\rho\overline{h}_l$ has a positive distance (independent of k) to any real root of $\varphi_{\tilde{\kappa}^{(l)}}$ of multiplicity larger than or equal to two, including roots lying on the coordinate axes. Notice that $\tilde{\kappa}_i^{(l)} < 1$ for any $l \in \{1, \dots, n\}$, $i \in \{1, 2\}$. In view of Lemma 2.2.4, we can conclude that on $\operatorname{supp} \nu\rho\overline{h}_l$ the Hessian of the polynomial $\varphi_{\tilde{\kappa}^{(l)}}$ does not vanish identically. Since $2^k R_{\tilde{\kappa}^{(l)}}(\delta_{2^{-k}}^{\tilde{\kappa}^{(l)}}(\cdot))$ is a perturbation term, the same holds true for

$$\varphi_{\tilde{\kappa}^{(l)}} + 2^k R_{\tilde{\kappa}^{(l)}}(\delta_{2^{-k}}^{\tilde{\kappa}^{(l)}}(\cdot)).$$

Remark 4.2.3. *To be very precise, we obtain that for every $x^0 \in \operatorname{supp} \nu\rho\overline{h}_l$ there exists a constant $C = C(x^0, 2^{-M}, \varepsilon_l, \varphi_{\tilde{\kappa}_l}) > 0$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $\alpha_1 + \alpha_2 = 2$ such that*

$$\left|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\varphi_{\tilde{\kappa}_l}(x^0)\right| \geq C.$$

Since $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\varphi_{\tilde{\kappa}_l}$ is continuous and $\operatorname{supp} \nu\rho\overline{h}_l$ is compact, we conclude that there is a constant $\tilde{C} > 0$ and finitely many open sets $U_1, \dots, U_K$ covering $\operatorname{supp} \nu\rho\overline{h}_l$ such that on each of these sets some second-order derivative of $\varphi_{\tilde{\kappa}_l}$ is bounded from below by $\tilde{C}$. If we assume Ω to be small enough, then the same holds true for $\varphi_{\tilde{\kappa}^{(l)}} + 2^k R_{\tilde{\kappa}^{(l)}}(\delta_{2^{-k}}^{\tilde{\kappa}^{(l)}}(\cdot))$ with $\frac{\tilde{C}}{2}$ instead of $\tilde{C}$. We can then decompose $\operatorname{supp} \nu\rho\overline{h}_l$ and correspondingly $\overline{\mathcal{M}}_{k,l}$ by means of a suitable partition of unity. In the sequel we shall use similar arguments without further precise description.

In order to obtain the desired result for $\overline{\mathcal{M}}_{k,l}$, we use again the results in [40]. The L^p-estimate for $\overline{\mathcal{M}}_l$ follows from Minkowski's inequality.
Next, we turn our attention to the maximal operators $\mathcal{M}_{\eta_{l,j}}$. Recall that in $\mathcal{H}$ the coordinates are localized to the narrow $\tilde{\kappa}^{(l)}$-homogeneous domain

$$\{(x_1, x_2) \in \mathcal{H} : \; |x_2 - z_{l,j}x_1^{s_l}| \leq 2\varepsilon_l x_1^{s_l}\}$$

near the origin. Observe that if each homogeneous polynomial $\varphi_{\tilde{\kappa}^{(l)}}$ does not contain real roots away from the coordinate axes with multiplicity higher than two, we would already finish the proof of Theorem 1.3.1. In order to proceed to the next step, we change variables

$$\zeta_{l,j}(x_1, x_2) = (x_1, x_2 - z_{l,j} x_1^{s_l}) .$$

Observe that $\zeta_{l,j}$ is a diffeomorphism on $\mathcal{H}$. Recall that $s_l = \frac{p_l}{q_l} = \frac{\tilde{\kappa}_2^{(l)}}{\tilde{\kappa}_1^{(l)}}$. We conclude

$$\mathcal{M}_{\eta_{l,j}} f(\cdot) = \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - t\Phi_{l,j}(x)) \psi(\zeta_{l,j}^{-1}(x)) \eta\left(\frac{x_2}{\varepsilon_l x_1^{s_l}}\right) dx, \tag{4.12}$$

where

$$\Phi_{l,j}(x_1, x_2) = (x_1, x_2 + z_{l,j} x_1^{s_l}, \varphi(x_1, x_2 + z_{l,j} x_1^{s_l})) , \quad (x_1, x_2) \in \mathcal{H}.$$

In view of Lemma 2.3.1 we can decompose

$$\varphi(x_1, x_2 + z_{l,j} x_1^{s_l}) = \varphi_{\kappa^{(l)}}(x_1^{\frac{1}{q_l}}, x_2) + R_{\kappa^{(l)}}(x_1^{\frac{1}{q_l}}, x_2).$$

The functions $\varphi_{\kappa^{(l)}}$ and $R_{\kappa^{(l)}}$ are analytic. As in Lemma 2.3.1

$$\kappa^{(l)} = \left(\kappa_1^{(l)}, \kappa_2^{(l)}\right) = \left(\frac{\tilde{\kappa}_1^{(l)}}{q_l}, \tilde{\kappa}_2^{(l)}\right) = \left(\frac{1}{m_l}, \frac{p_l}{m_l}\right).$$

More precisely, the function $\varphi_{\kappa^{(l)}}$ is a $\kappa^{(l)}$-homogeneous polynomial of degree one and $R_{\kappa^{(l)}}$ only contains terms of higher $\kappa^{(l)}$-degree. Furthermore, we know that (cf. Lemma 2.3.1)

$$\mathcal{N}_d(\varphi_{\kappa^{(l)}}) = [(q_l A_{l-1}, B_{l-1}), (P_l, n_j^{(l)})]$$

with some $P_l \geq q_l A_{l-1}$ and by assumption $n_j^{(l)} \geq 2$. The function $\psi \circ \zeta_{l,j}^{-1}$ is a smooth function on $\mathcal{H}$ and is supported in some small neighborhood of the origin. At this stage the preparation step is finished and we are left to estimate finitely many maximal operators (4.12).

Remark 4.2.4. *In the next section we shall describe the general algorithm how to proceed from the k-th step to the $(k+1)$-th step. For the arguments it will be important that the value s_l from the preparation step is not equal to one. Clearly, the Newton diagram of φ can contain an edge with slope one. In the case $s_l = 1$ we argue as follows. Observe that* $\gcd(p_l, q_l) = 1$, *and $p_l = q_l$ implies $p_l = q_l = 1$. In particular, $\zeta_{l,j}$ is just a linear* *transformation of coordinates. Let*

$$A(y_1, y_2, y_3) = (y_1, y_2 + z_{l,j}y_1, y_3).$$

It is obvious that $A \in \mathrm{GL}(3, \mathbb{R})$. *Then clearly,*

$$\Phi_{l,j}(x) = A(x_1, x_2, \underbrace{\varphi_{\kappa^{(l)}}(x_1, x_2) + R_{\kappa^{(l)}}(x_1, x_2)}_{\varphi_{l,j}(x_1,x_2)}).$$

In view of Lemma 2.2.1 we can just assume that $\Phi_{l,j}$ *is a parametrization of the graph of the analytic function* $\varphi_{l,j}$ *and use exactly the same arguments as before once again. Taking into account that the absolute value of the slope of each edge of the Newton diagram* $\mathcal{N}_d(\varphi_{l,j})$ *which lies below the line* $\{(t_1, t_2) : \ t_2 = n_j^{(l)}\}$ *(if at all existent) is strictly less than one, we eventually can assume that* $s_l \neq 1$.

4.3 Description of the l-th step, $l \geq 1$

We shall denote by I_l the index vector $I_l = (i_0, \ldots, i_l)$. Each entry i_k, $k \geq 1$, of the vector I_l corresponds to a certain quantity from the $(k-1)$-th step and varies in some finite range $\{1, \ldots, K_{i_{k-1}}\}$, where the integer $K_{i_{k-1}}$ depends on the number i_{k-1}. The preparation step, described in the previous section, is interpreted as the step number zero. The description below will explain this recursion. The index i_0 lies in the set $\{1, \ldots, n\}$, where n is the total number of edges from the preparation step. The entry i_1 lies in the subset of the index set $\{1, \ldots, M_{i_0}\}$ (we refer to (4.11)) of all non-trivial real zeros of $\varphi_{\tilde{\kappa}^{(i_0)}}$ with multiplicity greater than or equal to two.
From now we replace the isotropic dilation by the more general family of the non-isotropic dilations D_t^a defined in (1.3).
In the beginning of the l-th step, $l \geq 1$, we have to deal with finitely many maximal operators $\mathcal{M}_{I_l}$ of the form

$$\mathcal{M}_{I_l} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))\eta_{i_l}(x)dx, \quad f \geq 0.$$

The parametrization Φ_{i_l} is given by

$$\Phi_{i_l}(x_1, x_2) = \left(x_1, x_2 + z_{i_0,i_1} x_1^{s_{i_0}} + r_{i_l}(x_1), \varphi_{i_l}(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2) \right),$$

and

$$r_{i_l}(x_1) = z_{i_0,i_1,i_2} x_1^{s_{i_1}} + \ldots + z_{i_0,\ldots,i_l} x_1^{s_{i_{l-1}}},$$

with the interpretation $r_{i_1} = 0$. Each real number $z_{i_0,\ldots,i_{k+1}}$ was chosen in the k-th step, $k \geq 1$, in the way which will be described in detail later. From the preparation step it is known that the number z_{i_0,i_1} corresponds to some zero curve of $\varphi_{\tilde{\kappa}^{(i_0)}}$ in $\mathcal{H}$ which does not lie on the x_1-axis and with multiplicity greater than or equal to two. In particular, $z_{i_0,i_1} \neq 0$. The exponents s_{i_k} are rational and strictly increasing. More precisely, we have the inequality

$$1 \neq s_{i_0} = \frac{p_{i_0}}{q_{i_0}} < s_{i_1} = \frac{p_{i_1}}{q_{i_0} q_{i_1}} < \ldots < s_{i_{l-1}} = \frac{p_{i_{l-1}}}{q_{i_0} q_{i_1} \cdot \ldots \cdot q_{i_{l-1}}}. \tag{4.13}$$

For the assumption $s_{i_0} \neq 1$ we refer to Remark 4.2.4. We set $Q_{I_{l-1}} = \prod_{r=0}^{l-1} q_{i_r}$. The function φ_{i_l} is analytic and can be decomposed in the sense of (2.2) in two functions

$$\varphi_{i_l} = \varphi_{\kappa^{(i_{l-1})}, i_l} + R_{\kappa^{(i_{l-1})}, i_l},$$

where $\varphi_{\kappa^{(i_{l-1})}, i_l}$ is a $\kappa^{(i_{l-1})}$-homogeneous polynomial of degree one with

$$\kappa^{(i_{l-1})} = \left(\kappa_1^{(i_{l-1})}, \kappa_2^{(i_{l-1})} \right) = \left(\frac{1}{m_{i_{l-1}}}, \frac{p_{i_{l-1}}}{m_{i_{l-1}}} \right) \in \mathbb{Q}_+^2, \quad \gcd(p_{i_{l-1}}, m_{i_{l-1}}) = 1.$$

The function $R_{\kappa^{(i_{l-1})}, i_l}$ is analytic, consisting only of terms of higher $\kappa^{(i_{l-1})}$-degree. The Taylor support $\mathcal{T}(\varphi_{\kappa^{(i_{l-1})}, i_l})$ of $\varphi_{\kappa^{(i_{l-1})}, i_l}$, consisting of possible only one point, lies on the line

$$L_{\kappa^{(i_{l-1})}} = \left\{ (t_1, t_2) : \ \kappa_1^{(i_{l-1})} t_1 + \kappa_2^{(i_{l-1})} t_2 = 1 \right\}$$

with the absolute value of the slope $\frac{\kappa_1^{(i_{l-1})}}{\kappa_2^{(i_{l-1})}} = \frac{1}{p_{i_{l-1}}}$. The right endpoint of the Newton diagram $\mathcal{N}_d(\varphi_{\kappa^{(i_{l-1})}, i_l})$ is $(A_{i_l}, B_{i_l}) \in \mathbb{N}_0^2$, where $B_{i_l} \geq 2$, since otherwise the procedure would have stopped in the previous step, as we will see later. Recall that from the preparation step we have $B_{i_1} \geq 2$, since B_{i_1} is the multiplicity of the non-trivial real zero curve $\{(r, z_{i_0,i_1} r^{\frac{p_{i_0}}{q_{i_0}}}) : \ r > 0\}$ of $\varphi_{\tilde{\kappa}^{(i_0)}}$, i.e. $B_{i_1} = n_{i_1}^{(i_0)}$. The function η_{i_l} is given by

$$\eta_{i_l}(x_1, x_2) = \eta\left(\frac{x_2}{\varepsilon_{i_l} x_1^{s_{i_{l-1}}}}\right)\psi_{i_l}(x_1, x_2), \quad (x_1, x_2) \in \mathcal{H}, \tag{4.14}$$

where η is the positive bump function defined in (4.8) and ψ_{i_l} is a smooth positive function supported in a small neighborhood of the origin. More precisely,

$$\psi_{i_l}(x_1, x_2) = \psi(x_1, x_2 + z_{i_0,i_1} x_1^{s_{i_0}} + r_{i_l}(x_1)).$$

The dyadic parameter $\varepsilon_{i_l} = 2^{-M_{i_l}} > 0$, $M_{i_l} \in \mathbb{N}$, is assumed to be small enough.
We want to describe each stage of the l-th step and understand exactly under which circumstances we shall proceed to the next step.

4.3.1 Case 1: $\mathcal{N}(\varphi_{i_l}) \subseteq \{(t_1, t_2) : \ t_2 \geq B_{i_l}\}$

In other words, the Newton diagram $\mathcal{N}_d(\varphi_{i_l})$ does not contain any further edge below the vertex (A_{i_l}, B_{i_l}). In this case the estimate can be treated by means of the bi-dyadic decomposition without proceeding to the next step. Thus we obtain the pointwise estimate

$$\begin{aligned} \mathcal{M}_{I_l} f(\cdot) &\leq \sum_{j,k=N}^{\infty} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))\eta_{i_l}(x)\chi_j(x_1)\chi_k(x_2)dx \\ &= \sum_{j,k=N}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(2^{-j}x_1, 2^{-k}x_2)))\eta_{i_l}(2^{-j}x_1, 2^{-k}x_2)\chi \otimes \chi(x)dx, \end{aligned}$$

with $\chi \otimes \chi$ defined in (4.5). By (4.6) we have

$$\operatorname{supp} \chi \otimes \chi \subseteq \left\{x_1 : \ 2^{-1} \leq |x_1| \leq 2\right\} \times \left\{x_2 : \ 2^{-1} \leq |x_2| \leq 2\right\}.$$

Observe that

$$\eta_{i_l}(2^{-j}x_1, 2^{-k}x_2) = \eta\left(\frac{2^{-k}x_2}{\varepsilon_{i_l} 2^{-js_{i_{l-1}}} x_1^{s_{i_{l-1}}}}\right)\psi_{i_l}(2^{-j}x_1, 2^{-k}x_2).$$

Thus for any pair (j,k) and any $x \in \mathcal{H}$ with $\eta_{i_l}(2^{-j}x_1, 2^{-k}x_2)\chi \otimes \chi(x) \neq 0$, we conclude

$$2^{-k-1} \leq 2^{-k}|x_2| \leq 2\varepsilon_{i_l} 2^{-j\frac{p_{i_{l-1}}}{Q_{I_{l-1}}}} x_1^{\frac{p_{i_{l-1}}}{Q_{I_{l-1}}}} \leq 2 \cdot 2^{-M_{i_l}} 2^{\frac{p_{i_{l-1}}}{Q_{I_{l-1}}}} 2^{-j\frac{p_{i_{l-1}}}{Q_{I_{l-1}}}},$$

and therefore

$$2^{-k+j\frac{p_{i_{l-1}}}{Q_{I_{l-1}}}} \leq 2^{\frac{p_{i_{l-1}}}{Q_{I_{l-1}}}+2-M_{i_l}} \leq 2^{-\frac{M_{i_l}}{2}}, \tag{4.15}$$

since M_{i_l} was assumed to be a sufficiently large positive integer. Next, observe that

$$\Phi_{i_l}(2^{-j}x_1, 2^{-k}x_2) = \begin{pmatrix} 2^{-j}x_1 \\ 2^{-js_{i_0}}(2^{-k+js_{i_0}}x_2 + z_{i_0,i_1}x_1^{s_{i_0}} + 2^{js_{i_0}}r_{i_l}(2^{-j}x_1)) \\ \varphi_{i_l}((2^{-j}x_1)^{\frac{1}{Q_{I_{l-1}}}}, 2^{-k}x_2) \end{pmatrix}^T,$$

and that

$$\varphi_{i_l}((2^{-j}x_1)^{\frac{1}{Q_{I_{l-1}}}}, 2^{-k}x_2) = 2^{-j\frac{A_{i_l}}{Q_{I_{l-1}}}-kB_{i_l}} \left(C_{A_{i_l},B_{i_l}} x_1^{\frac{A_{i_l}}{Q_{I_{l-1}}}} x_2^{B_{i_l}} + P_{i_l}^{j,k}(x_1,x_2) \right), \; C_{A_{i_l},B_{i_l}} \neq 0,$$

where exactly as in the previous section we see using (4.15) and

$$\mathcal{N}(\varphi_{i_l}) \subseteq \{(t_1,t_2) : \; t_2 \geq B_{i_l}\}$$

that $P_{i_l}^{j,k}$ is a perturbation term. Clearly, $2^{js_{i_0}}r_{i_l}(2^{-j}\cdot)$ is a perturbation term in $2^{-\frac{j}{L}}$ for some $L \in \mathbb{N}$, because of (4.13). More precisely, for some $L \in \mathbb{N}$

$$2^{js_{i_0}}r_{i_l}(2^{-j}\cdot) = \tilde{r}_{i_l}(x_1, 2^{-\frac{j}{L}}),$$

where $\tilde{r}_{i_l}$ is smooth and we have $\partial_{x_1}^B \tilde{r}_{i_l}(x_1,0) = 0$ for every $(x_1,B) \in (\frac{1}{4},4) \times \mathbb{N}_0$. Thus if we apply again Lemma 2.2.1, we see that it is sufficient to show that each operator

$$\mathcal{M}_{I_l}^{(j,k)} f(\cdot) = 2^{-j-k} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D_t^a(\Phi_{i_l}^{(j,k)}(x)))\chi \otimes \chi(x)\nu(x)dx, \quad j,k \geq N,$$

with

$$\Phi_{i_l}^{(j,k)}(x_1,x_2) = \begin{pmatrix} x_1 \\ 2^{-k+js_{i_0}}x_2 + z_{i_0,i_1}x_1^{s_{i_0}} + 2^{js_{i_0}}r_{i_l}(2^{-j}x_1) \\ C_{A_{i_l},B_{i_l}}x_1^{\frac{A_{i_l}}{Q_{I_{l-1}}}}x_2^{B_{i_l}} + P_{i_l}^{j,k}(x_1,x_2)) \end{pmatrix},$$

is bounded on $L^2(\mathbb{R}^3)$ with the norm at most a constant multiple of $2^{-\frac{k}{4}-j}$. The function ν is a smooth positive bump function supported in $\mathcal{H}$ and is identically one on $\operatorname{supp}\chi\otimes\chi\cap\mathcal{H}$. Since $B_{i_l}\geq 2$, on $\operatorname{supp}\chi\otimes\chi\cap\mathcal{H}$ we have

$$\left|\partial_2 C_{A_{i_l},B_{i_l}}x_1^{\frac{A_{i_l}}{Q_{I_{l-1}}}}x_2^{B_{i_l}}\right| \sim 1 \sim \left|\partial_2^2 C_{A_{i_l},B_{i_l}}x_1^{\frac{A_{i_l}}{Q_{I_{l-1}}}}x_2^{B_{i_l}}\right|.$$

Since $s_{i_0}\neq 1$, on $\pi_1(\operatorname{supp}\chi\otimes\chi\cap\mathcal{H})$ we have

$$\left|\partial_1^2 z_{i_0,i_1}x_1^{s_{i_0}}\right| \sim 1.$$

Lemma 3.3.2 implies that the corresponding oscillatory integral

$$\Lambda_{I_l}^{(j,k)}(\lambda) = 2^{-j-k}\int\limits_{\mathbb{R}^2} e^{-i\lambda\cdot\Phi_{i_l}^{(j,k)}(x)}\chi\otimes\chi(x)\nu(x)dx, \quad \lambda\in\mathbb{R}^3,$$

can be estimated by

$$\begin{aligned} \left|\Lambda_{I_l}^{(j,k)}(\lambda)\right| + \left|\nabla\Lambda_{I_l}^{(j,k)}(\lambda)\right| &\lesssim \min\left\{2^{-j-k}, \frac{2^{-j-k}}{2^{-k+js_{i_0}}(1+|\lambda|)}\right\} \\ &\lesssim 2^{-j}\min\left\{2^{-k}, \frac{1}{(1+|\lambda|)}\right\}, \end{aligned}$$

if the integer N is large enough, i.e. j, k are large.
Taking a geometric mean we obtain

$$\left|\Lambda_{I_l}^{(j,k)}(\lambda)\right| + \left|\nabla\Lambda_{I_l}^{(j,k)}(\lambda)\right| \lesssim \frac{2^{-j}2^{-\frac{k}{4}}}{(1+|\lambda|)^{\frac{3}{4}}}.$$

Theorem 4.0.13 gives $\|\mathcal{M}_{I_l}^{(j,k)}\|_{L^2\to L^2}\lesssim 2^{-j-\frac{k}{4}}$. The desired result follows from Minkowski's inequality.

Remark 4.3.1. *In the sequel we will not write down explicitly the Fourier transform of the corresponding convolution kernel of the maximal operator as the oscillatory integral and only refer to it as the corresponding or the associated oscillatory integral.*

Eventually, we conclude that the procedure terminates at this stage if $\mathcal{T}(\varphi_{i_l})$ does not contain further points below the line $t_2 = B_{i_l}$.

4.3.2 Case 2: $\mathcal{N}(\varphi_{i_l}) \not\subseteq \{(t_1, t_2) : \ t_2 \geq B_{i_l}\}$

This means $\mathcal{N}_d(\varphi_{i_l})$ contains further vertices lying below the line $t_2 = B_{i_l}$. Denote the closest vertex of $\mathcal{N}_d(\varphi_{i_l})$ lying below the line $t_2 = B_{i_l}$ by $(\tilde{A}_{i_l}, \tilde{B}_{i_l})$. In particular, $\tilde{B}_{i_l} < B_{i_l}$. The edge $[(A_{i_l}, B_{i_l}), (\tilde{A}_{i_l}, \tilde{B}_{i_l})]$ lies on the line

$$L_{\tilde{\kappa}^{(i_l)}} = \left\{(t_1, t_2) : \ \tilde{\kappa}_1^{(i_l)} t_1 + \tilde{\kappa}_2^{(i_l)} t_2 = 1\right\},$$

with the uniquely determined weight $\tilde{\kappa}^{(i_l)}$. More precisely, we can write

$$\tilde{\kappa}^{(i_l)} = \left(\tilde{\kappa}_1^{(i_l)}, \tilde{\kappa}_2^{(i_l)}\right) = \left(\frac{q_{i_l}}{m_{i_l}}, \frac{p_{i_l}}{m_{i_l}}\right), \quad \gcd(p_{i_l}, q_{i_l}) = 1.$$

Notice also that the absolute value of the slope of the edge $[(A_{i_l}, B_{i_l}), (\tilde{A}_{i_l}, \tilde{B}_{i_l})]$ is strictly less than $\frac{\kappa_1^{(i_{l-1})}}{\kappa_2^{(i_{l-1})}}$, i.e.

$$\frac{1}{p_{i_l}} \leq \frac{q_{i_l}}{p_{i_l}} = \frac{\tilde{\kappa}_1^{(i_l)}}{\tilde{\kappa}_2^{(i_l)}} < \frac{\kappa_1^{(i_{l-1})}}{\kappa_2^{(i_{l-1})}} = \frac{1}{p_{i_{l-1}}}.$$

Similar to the preparation step, we split the domain of the integration into a homogeneous domain, corresponding to the above edge and the transition domain between both of the different homogeneities $\kappa^{(i_{l-1})}$ and $\tilde{\kappa}^{(i_l)}$. To this end, for a large number $N_{i_l} \in \mathbb{N}$ we write

$$\eta_{i_l} = \tau_{i_l} + \tilde{\eta}_{i_l},$$

where

$$\tau_{i_l}(x_1, x_2) = \eta\left(\frac{x_2}{\varepsilon_{i_l} x_1^{s_{i_{l-1}}}}\right)\left(1 - \eta\left(\frac{x_2}{N_{i_l} x_1^{s_{i_l}}}\right)\right)\psi_{i_l}(x_1, x_2), \quad (x_1, x_2) \in \mathcal{H},$$

$$\tilde{\eta}_{i_l}(x_1,x_2) = \eta\left(\frac{x_2}{\varepsilon_{i_l} x_1^{s_{i_{l-1}}}}\right)\eta\left(\frac{x_2}{N_{i_l} x_1^{s_{i_l}}}\right)\psi_{i_l}(x_1,x_2), \quad (x_1,x_2)\in\mathcal{H},$$

and

$$s_{i_l} = \frac{p_{i_l}}{q_{i_0}q_{i_1}\cdot\ldots\cdot q_{i_l}} = \frac{p_{i_l}}{Q_{I_l}} > s_{i_{l-1}},$$

as in (4.13). Observe that in $\mathcal{H}$ the function τ_{i_l} is supported in the transition domain

$$\left\{(x_1,x_2)\in\mathcal{H} : N_{i_l} x_1^{s_{i_l}} \le |x_2| \le 2\varepsilon_{i_l} x_1^{s_{i_{l-1}}}\right\}\cap \operatorname{supp}\psi_{i_l}.$$

In $\mathcal{H}$ the function $\tilde{\eta}_{i_l}$ is supported in

$$S_{i_l} = \{(x_1,x_2)\in\mathcal{H}\ : |x_2|\le 2\varepsilon_{i_l}x_1^{s_{i_{l-1}}}\}\cap\{(x_1,x_2)\in\mathcal{H}\ : |x_2|\le 2N_{i_l}x_1^{s_{i_l}}\}\cap\operatorname{supp}\psi_{i_l}.$$

Since we can assume that $\operatorname{supp}\psi_{i_l}$ is small enough, then ε_{i_l} can be chosen sufficiently small and the integer N_{i_l} sufficiently large such that

$$S_{i_l} = \left\{(x_1,x_2)\in\mathcal{H}\ : |x_2|\le 2N_{i_l}x_1^{s_{i_l}}\right\}\cap\operatorname{supp}\psi_{i_l},$$

and

$$\tilde{\eta}_{i_l}(x_1,x_2) = \eta\left(\frac{x_2}{N_{i_l}x_1^{s_{i_l}}}\right)\psi_{i_l}(x_1,x_2).$$

We therefore can estimate $\mathcal{M}_{I_l}f$ pointwise by the sum of two maximal operators

$$\mathcal{M}_{I_l}^{\tau_{i_l}}f(\cdot) = \sup_{t>0}\int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))\tau_{i_l}(x)dx,$$

$$\mathcal{M}_{I_l}^{\tilde{\eta}_{i_l}}f(\cdot) = \sup_{t>0}\int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))\tilde{\eta}_{i_l}(x)dx.$$

If we decompose again bi-dyadically as in the previous case and use the same arguments, we see that the maximal operator $\mathcal{M}_{I_l}^{\tau_{i_l}}$ is bounded on $L^2(\mathbb{R}^3)$. The domain of integration of the maximal operator $\mathcal{M}_{I_l}^{\tilde{\eta}_{i_l}}$ requires further analysis. We decompose as in (2.2)

$$\varphi_{i_l} = \varphi_{\tilde{\kappa}^{(i_l)}} + R_{\tilde{\kappa}^{(i_l)}},$$

where $\varphi_{\tilde{\kappa}^{(i_l)}}$ is the $\tilde{\kappa}^{(i_l)}$-homogeneous part of degree one of the function φ_{i_l} and

$$R_{\tilde{\kappa}^{(i_l)}} = \varphi_{i_l} - \varphi_{\tilde{\kappa}^{(i_l)}}$$

consists of terms of higher $\tilde{\kappa}^{(i_l)}$-degree. The singular points in the domain $\mathcal{H} \cap \operatorname{supp} \tilde{\eta}_{i_l}$ are the zeros of the polynomial in fractional power $\partial_2 \varphi_{\tilde{\kappa}^{(i_l)}} \left(\cdot^{\frac{1}{Q_{I_{l-1}}}}, \cdot \right)$. Away from those zeros the described procedure terminates, since after a dyadic decomposition the corresponding oscillatory integral can be estimated appropriately. Since $\partial_2 \varphi_{\tilde{\kappa}^{(i_l)}}$ is also a $\tilde{\kappa}^{(i_l)}$-homogeneous polynomial, the zeros of $\partial_2 \varphi_{\tilde{\kappa}^{(i_l)}}$ in $\mathcal{H}$ (if at all existent) are finitely many curves of the form

$$\mathcal{C}_{I_l,\alpha} = \left\{ \left(r, z_{I_l,\alpha} r^{\frac{p_{i_l}}{q_{i_l}}} \right) : \ r > 0 \right\}, \quad z_{I_l,\alpha} \in \mathbb{R}.$$

The index α varies in some finite range depending on i_l. Observe that we have the identity

$$\frac{\tilde{\kappa}_2^{(i_l)}}{\tilde{\kappa}_1^{(i_l)}} = \frac{p_{i_l}}{q_{i_l}} = s_{i_l} \prod_{r=0}^{l-1} q_{i_r} = s_{i_l} Q_{I_{l-1}}.$$

We proceed as follows. For each α we fix a sufficiently small dyadic number $\varepsilon_\alpha = 2^{-M_\alpha}$, $M_\alpha \in \mathbb{N}$, and localize the coordinates to the narrow $(\tilde{\kappa}_1^{(i_l)} Q_{I_{l-1}}, \tilde{\kappa}_2^{(i_l)})$-homogeneous domains near the singularities of $\partial_2 \varphi_{\tilde{\kappa}^{(i_l)}} \left(\cdot^{\frac{1}{Q_{I_{l-1}}}}, \cdot \right)$ in $\mathcal{H}$ by means of

$$\eta_\alpha(x_1, x_2) = \eta \left(\frac{x_2 - z_{I_l,\alpha} x_1^{s_{i_l}}}{\varepsilon_\alpha x_1^{s_{i_l}}} \right) \psi_{i_l}(x_1, x_2).$$

Observe that if each ε_α is chosen sufficiently small, then the domains $\operatorname{supp} \eta_\alpha \cap \mathcal{H}$ are disjoint in α. We get

$$\tilde{\eta}_{i_l} \quad = \quad \tilde{\eta}_{i_l} \sum_\alpha \eta_\alpha + \left(1 - \sum_\alpha \eta_\alpha \right) \tilde{\eta}_{i_l}.$$

Notice that if N_{i_l} is chosen sufficiently large, e.g.

$$N_{i_l} \geq 3 \max_\alpha \{ |z_{I_l,\alpha}| \} + 6,$$

then clearly,

$$\tilde{\eta}_{i_l} \eta_\alpha = \eta_\alpha \quad \text{in } \mathcal{H} \text{ for every } \alpha.$$

Estimate the maximal operator $\mathcal{M}_{I_l}^{\tilde{\eta}_{i_l}} f$ by

$$\mathcal{M}_{I_l}^{\tilde{\eta}_{i_l}} f \leq \overline{\mathcal{M}}_{I_l} f + \sum_{\alpha} \mathcal{M}_{I_l,\alpha} f,$$

where

$$\mathcal{M}_{I_l,\alpha} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))\eta_\alpha(x)dx, \tag{4.16}$$

$$\overline{\mathcal{M}}_{I_l} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))(1 - \sum_{\alpha} \eta_\alpha(x))\tilde{\eta}_{i_l}(x)dx.$$

The procedure does not stop in this step for the operators $\mathcal{M}_{I_l,\alpha}$ and we proceed to the $(l+1)$-th step or to the stopping time algorithm. First, we show that the maximal operator $\overline{\mathcal{M}}_{I_l}$ is bounded on $L^2(\mathbb{R}^3)$. To this end we decompose the maximal operator into dyadic pieces. Consider the dilations adapted to $\varphi_{\tilde{\kappa}^{(i_l)}}\left(\cdot^{\frac{1}{Q_{I_{l-1}}}}, \cdot\right)$

$$\delta_r(x_1, x_2) = \left(r^{Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} x_1, r^{\tilde{\kappa}_2^{(i_l)}} x_2\right), \quad r > 0.$$

Then for any $x \in \mathcal{H}$ and $r > 0$ we get

$$\varphi_{\tilde{\kappa}^{(i_l)}}\left(\delta_r\left(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2\right)\right) = r\varphi_{\tilde{\kappa}^{(i_l)}}\left(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2\right).$$

As in Lemma A.0.2 we write

$$\sum_{k=N}^{\infty} \rho_k(x) = 1 \quad \text{for } x \in \operatorname{supp} \psi_{i_l} \setminus \{0\}.$$

Here is $\rho_k(x) = \rho(\delta_{2^k}(x))$. We can assume N to be sufficiently large. We get the pointwise estimate

$$\begin{aligned}
\overline{\mathcal{M}}_{I_l} f(\cdot) &\leq \sum_{k=N}^{\infty} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_l}(x)))(1 - \sum_{\alpha} \eta_\alpha(x))\tilde{\eta}_{i_l}(x)\rho_k(x)dx \\
&= \sum_{k=N}^{\infty} 2^{-k(\tilde{\kappa}_1^{(i_l)}Q_{I_{l-1}} + \tilde{\kappa}_2^{(i_l)})} \overline{\mathcal{M}}_{I_l}^k f(\cdot),
\end{aligned}$$

where we set

$$\overline{\mathcal{M}}^k_{I_l} f(\cdot) = \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D^a_t(\Phi_{i_l}(\delta_{2^{-k}}(x)))(1 - \sum_{\alpha} \eta_\alpha(\delta_{2^{-k}}(x)))\tilde{\eta}_{i_l}(\delta_{2^{-k}}(x))\rho(x)dx.$$

Notice that

$$\operatorname{supp}\Big((1 - \sum_{\alpha} \eta_\alpha(\delta_{2^{-k}}(\cdot)))\tilde{\eta}_{i_l}(\delta_{2^{-k}}(\cdot))\rho\Big) \subseteq \{x \in \mathbb{R}^2 : \ 1 \leq |x| \leq R\}$$

for some R only depending on $(\tilde{\kappa}_1^{(i_l)} Q_{I_{l-1}}, \tilde{\kappa}_2^{(i_l)})$. Furthermore, observe that the support has a positive distance, independent of k, to each zero curve of $\partial_2 \varphi_{\tilde{\kappa}^{(i_l)}}\left(\cdot^{\frac{1}{Q_{I_{l-1}}}}, \cdot\right)$ in $\mathcal{H}$ and that in $\mathcal{H} \cap \operatorname{supp}\Big((1 - \sum_{\alpha} \eta_\alpha(\delta_{2^{-k}}(\cdot)))\tilde{\eta}_{i_l}(\delta_{2^{-k}}(\cdot))\rho\Big)$ we also have $x_1 \sim 1$ and $|x_2| \lesssim 1$. Next, observe that $\Phi_{i_l}(\delta_{2^{-k}}(x))$ is equal to

$$\begin{pmatrix} 2^{-kQ_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} x_1 \\ 2^{-ks_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}}(2^{-k\tilde{\kappa}_2^{(i_l)}+ks_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} x_2 + z_{i_0,i_1} x_1^{s_{i_0}} + 2^{ks_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} r_{i_l}(2^{-kQ_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} x_1)) \\ 2^{-k}\left(\varphi_{\tilde{\kappa}^{(i_l)}}(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2) + 2^k R_{\tilde{\kappa}^{(i_l)}}(\delta_{2^{-k}}(x_1^{\frac{1}{Q_{I_{l-1}}}}), x_2)\right) \end{pmatrix}^T.$$

Again, applying Lemma 2.2.1 we see that we only need to analyze the sum of the maximal operators

$$\mathcal{M}^k_{I_l} f(\cdot) = 2^{-k(\tilde{\kappa}_1^{(i_l)}Q_{I_{l-1}}+\tilde{\kappa}_2^{(i_l)})} \sup_{t>0} \int\limits_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi^k_{i_l}(x)))((1 - \sum_{\alpha} \eta_\alpha)\tilde{\eta}_{i_l})(\delta_{2^{-k}}(x))\mu(x)\rho(x)dx,$$

where

$$\Phi^k_{i_l}(x_1, x_2) = \begin{pmatrix} x_1 \\ 2^{-k(\tilde{\kappa}_2^{(i_l)} - s_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)})} x_2 + z_{i_0,i_1} x_1^{s_{i_0}} + 2^{ks_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} r_{i_l}(2^{-kQ_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} x_1) \\ \varphi_{\tilde{\kappa}^{(i_l)}}(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2) + 2^k R_{\tilde{\kappa}^{(i_l)}}(\delta_{2^{-k}}(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2)) \end{pmatrix}^T,$$

and μ is a suitable positive bump function supported in $\mathcal{H}$ and identically one on $\mathcal{H} \cap \operatorname{supp}\Big((1 - \sum_{\alpha} \eta_\alpha(\delta_{2^{-k}}(\cdot)))\tilde{\eta}_{i_l}(\delta_{2^{-k}}(\cdot))\rho\Big)$. It is not difficult to see that the functions

$2^k R_{\tilde{\kappa}^{(i_l)}}(\delta_{2^{-k}}\cdot)$ and $2^{ks_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}} r_{i_l}(2^{-kQ_{I_{l-1}}\tilde{\kappa}_1^{(i_l)}}\cdot)$ are small perturbations.
We see that $\tilde{\kappa}_2^{(i_l)} - s_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)} > 0$ is equivalent to

$$\frac{p_{i_l}}{q_{i_l}Q_{I_{l-1}}} > s_{i_0},$$

which is true due to (4.13), and therefore the coefficient $2^{-k(\tilde{\kappa}_2^{(i_l)} - s_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)})}$ is a very small parameter. Since $1 \neq s_{i_0} \neq 0$ and $z_{i_0,i_1} \neq 0$, we also conclude that

$$\min\left\{|\partial_1(z_{i_0,i_1}x_1^{s_{i_0}}), |\partial_1^2(z_{i_0,i_1}x_1^{s_{i_0}})|\right\} \sim 1$$

on $\pi_1\left(\mathcal{H} \cap \operatorname{supp}\left((1 - \sum_\alpha \eta_\alpha(\delta_{2^{-k}}(\cdot)))\tilde{\eta}_{i_l}(\delta_{2^{-k}}(\cdot))\rho\right)\right)$. Furthermore, we have

$$\left|\partial_2^{B_{i_l}}\varphi_{\tilde{\kappa}^{(i_l)}}(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2)\right| \sim 1.$$

Recall that $B_{i_l} \geq 2$. Thus we see that the corresponding oscillatory integral $\Lambda_{I_l}^k$ satisfies the assumptions of Lemma 3.3.4. We obtain

$$\left|\Lambda_{I_l}^k(\lambda)\right| + \left|\nabla\Lambda_{I_l}^k(\lambda)\right| \lesssim 2^{-k(\tilde{\kappa}_1^{(i_l)}Q_{I_{l-1}} + \tilde{\kappa}_2^{(i_l)})} \cdot \frac{2^{k(\tilde{\kappa}_2^{(i_l)} - s_{i_0}Q_{I_{l-1}}\tilde{\kappa}_1^{(i_l)})}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B_{i_l}}}}.$$

uniformly in $k \geq N$, if N is large. We conclude

$$\left|\Lambda_{I_l}^k(\lambda)\right| + \left|\nabla\Lambda_{I_l}^k(\lambda)\right| \lesssim \frac{2^{-k\tilde{\kappa}_1^{(i_l)}Q_{I_{l-1}}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B_{i_l}}}}.$$

With Theorem 4.0.13 we conclude that $\|\mathcal{M}_{I_l}^k\|_{L^2\to L^2} \lesssim 2^{-k\tilde{\kappa}_1^{(i_l)}Q_{I_{l-1}}}$. The desired L^2-boundedness of $\overline{\mathcal{M}}_{I_l}$ follows from Minkowski's inequality. Thus we are only left with finitely many maximal operators $\mathcal{M}_{I_l,\alpha}$ from (4.16).
Here the procedure does not stop and we shall change variables, in order to proceed to the next step or to the stopping time algorithm. Consider the change of variables

$$\zeta_{I_l,\alpha}(x_1,x_2) = (x_1, x_2 - z_{I_l,\alpha}x_1^{\frac{p_{i_l}}{Q_{I_l}}}) = (x_1, x_2 - z_{I_l,\alpha}x_1^{s_{i_l}}), \quad (x_1,x_2) \in \mathcal{H}.$$

We get

$$\Phi_{i_l}(\zeta_{I_l,\alpha}^{-1}(x_1,x_2)) = \begin{pmatrix} x_1 \\ x_2 + z_{i_0,i_1}x_1^{s_{i_0}} + r_{i_l}(x_1) + z_{I_l,\alpha}x_1^{s_{i_l}} \\ \varphi_{i_l}\left(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2 + z_{I_l,\alpha}x_1^{\frac{p_{i_l}}{Q_{I_l}}}\right) \end{pmatrix}^T.$$

In view of Lemma 2.3.1 we see that there exists an analytic function φ_α such that

$$\varphi_{i_l}\left(x_1^{\frac{1}{Q_{I_{l-1}}}}, x_2 + z_{I_l,\alpha}x_1^{\frac{p_{i_l}}{Q_{I_l}}}\right) = \varphi_\alpha\left(x_1^{\frac{1}{Q_{I_l}}}, x_2\right).$$

Moreover, φ_α can be decomposed in

$$\varphi_\alpha(x_1,x_2) = \varphi_{\kappa^{(i_l)},\alpha}(x_1,x_2) + R_{\kappa^{(i_l)},\alpha}(x_1,x_2)$$

with usual notations

$$\kappa^{(i_l)} = (\kappa_1^{(i_l)}, \kappa_2^{(i_l)}) = \left(\frac{\tilde{\kappa}_1^{(i_l)}}{q_{i_l}}, \tilde{\kappa}_2^{(i_l)}\right) = \left(\frac{1}{m_{i_l}}, \frac{p_{i_l}}{m_{i_l}}\right).$$

The polynomial $\varphi_{\kappa^{(i_l)},\alpha}$ is $\kappa^{(i_l)}$-homogeneous of degree one. Its Taylor support lies on the line

$$L_{\kappa^{(i_l)}} = \left\{(t_1,t_2):\ \kappa_1^{(i_l)}t_1 + \kappa_2^{(i_l)}t_2 = 1\right\}.$$

The function $R_{\kappa^{(i_l)},\alpha}$ is analytic and its Taylor support lies above the line $L_{\kappa^{(i_l)}}$.
From Lemma 2.3.1 also follows that the left upper point of the face $\mathcal{N}_d(\varphi_{\kappa^{(i_l)},\alpha})$ is $(q_{i_l}A_{i_l}, B_{i_l}) \in \mathbb{N}_0^2$. Depending on whether or not the polynomial $\varphi_{\tilde{\kappa}^{(i_l)}}$ vanishes along the curve $\mathcal{C}_{I_l,\alpha}$, we proceed to the $(l+1)$-th step or end at this stage of the l-th step. Assume first that $\varphi_{\tilde{\kappa}^{(i_l)}}$ does not vanish along the curve $\mathcal{C}_{I_l,\alpha}$. This case leads to the situation of stopping time.

4.3.3 Auxiliary statements for the stopping time procedure

First, observe that Lemma 2.3.1 also reveals that in this case the Newton diagram of $\varphi_{\kappa^{(i_l)},\alpha}$ is given by

$$\mathcal{N}_d(\varphi_{\kappa^{(i_l)},\alpha}) = [(q_{i_l}A_{i_l}, B_{i_l}), (m_{i_l}, 0)].$$

In the next lemma we shall first establish some relevant relations for the quantities p_{i_l}, Q_{I_l} and m_{i_l}.

Lemma 4.3.2. *For each $l \geq 1$ we have*

$$p_{i_0}\prod_{r=1}^{l} q_{i_r} < p_{i_l}, \tag{4.17}$$

$$p_{i_0}\prod_{r=1}^{l} q_{i_r} < \frac{m_{i_l}}{2}, \tag{4.18}$$

$$Q_{I_l} < m_{i_l}. \tag{4.19}$$

Proof. The estimate (4.17) is easily proved by induction using

$$\frac{q_{i_l}}{p_{i_l}} < \frac{1}{p_{i_{l-1}}}.$$

In order to prove (4.18), recall first that for each $l \geq 1$ we have $B_{i_l} \geq 2$. Therefore we get

$$1 = \tilde{\kappa}_1^{(i_l)} A_{i_l} + \tilde{\kappa}_2^{(i_l)} B_{i_l} \geq 2\tilde{\kappa}_2^{(i_l)} = 2\kappa_2^{(i_l)},$$

which implies $\kappa_2^{(i_l)} \leq \frac{1}{2}$.
We conclude

$$\begin{aligned} p_{i_0} q_{i_1} \dots q_{i_{l-1}} \frac{q_{i_l}}{m_{i_l}} &= p_{i_0} q_{i_1} \dots q_{i_{l-1}} \frac{q_{i_l}}{p_{i_l}} \frac{p_{i_l}}{m_{i_l}} \\ &= p_{i_0} q_{i_1} \dots q_{i_{l-1}} \frac{q_{i_l}}{p_{i_l}} \kappa_2^{(i_l)} \\ &\leq \frac{1}{2} p_{i_0} q_{i_1} \dots q_{i_{l-1}} \frac{q_{i_l}}{p_{i_l}} \\ &\leq \frac{1}{2}, \end{aligned}$$

where in the last inequality we used (4.17).
In order to prove (4.19), recall first that at the beginning we assumed that $\nabla\varphi(0,0) = (0,0)$. In particular, $(1,0)$ does not belong to $\mathcal{T}(\varphi)$. This implies

$$1 < \frac{1}{\tilde{\kappa}_1^{(i_0)}} = \frac{m_{i_0}}{q_{i_0}},$$

and therefore $m_{i_0} > q_{i_0}$. Furthermore,

$$m_{i_0} < \frac{1}{\tilde{\kappa}_1^{(i_1)}} = \frac{m_{i_1}}{q_{i_1}} \leq m_{i_1},$$

which implies

$$m_{i_1} > m_{i_0} q_{i_1} > q_{i_0} q_{i_1}.$$

The rest of the proof is done by the induction using the simple observation that

$$\frac{m_{i_{l-1}}}{q_{i_{l-1}}} \leq m_{i_{l-1}} < \frac{1}{\tilde{\kappa}_1^{(i_l)}} = \frac{m_{i_l}}{q_{i_l}} \leq m_{i_l}.$$

□

Set $n_{i_l} = p_{i_0} \prod_{r=1}^{l} q_{i_r}$. We see that

$$x_2 + z_{i_0,i_1} x_1^{s_{i_0}} + \ldots + z_{i_0,\ldots,i_l} x^{s_{i_{l-1}}} + z_{I_l,\alpha} x^{s_{i_l}} = x_2 + P_{i_l}(x_1^{\frac{1}{Q_{I_l}}}),$$

where P_{i_l} is a polynomial of the form

$$P_{i_l}(x_1) = z_{i_0,i_1} x_1^{n_{i_l}} + o(x_1^{n_{i_l}}).$$

Clearly, $Q_{I_l} \neq n_{i_l}$, since $1 \neq s_{i_0} = \frac{p_{i_0}}{q_{i_0}} = \frac{n_{i_l}}{Q_{I_l}}$. With estimate (4.18) we see that $n_{i_l} < \frac{m_{i_l}}{2} < m_{i_l}$. Furthermore, since

$$\mathcal{N}_d(\varphi_{\kappa^{(i_l)},\alpha}) = [(q_{i_l} A_{i_l}, B_{i_l}), (m_{i_l}, 0)]$$

and $[(q_{i_l} A_{i_l}, B_{i_l}), (m_{i_l}, 0)] \cap \{(t_1, t_2) : t_2 = 1\} \cap \mathcal{T}(\varphi_{\kappa^{(i_l)},\alpha}) = \emptyset$ by Lemma 2.3.1, we conclude that

$$\varphi_{\kappa^{(i_l)},\alpha}(x_1, x_2) = c_1 x_1^{m_{i_l}} + x_2^B \left(c_2 x_1^A + \ldots + c_r x_1^{q_{i_l} A_{i_l}} x_2^{B_{i_l} - B} \right), \quad 2 \leq B \leq B_{i_l},$$

and c_1, $c_2 \in \mathbb{R} \setminus \{0\}$. By (4.19) we have $Q_{I_l} < m_{i_l}$. The estimate (4.17) implies $n_{i_l} < p_{i_l}$.

Eventually, in this case we see that we have to deal with the maximal operator

$$\mathcal{M}_{ST}f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))\eta\left(\frac{x_2}{\varepsilon_\alpha x_1^{s_{i_l}}}\right)\psi_{i_l}\left(x_1, x_2 + z_{I_l,\alpha}x_1^{s_{i_l}}\right)dx, \quad f \geq 0,$$

and

$$\Phi(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 + P_{i_l}\left(x_1^{\frac{1}{Q_{I_l}}}\right) \\ \varphi_{\kappa^{(i_l)},\alpha}\left(x_1^{\frac{1}{Q_{I_l}}}, x_2\right) + R_{\kappa^{(i_l)},\alpha}\left(x_1^{\frac{1}{Q_{I_l}}}, x_2\right) \end{pmatrix}^T.$$

The function

$$\mathcal{H} \ni (x_1, x_2) \longmapsto \psi_{i_l}\left(x_1, x_2 + z_{I_l,\alpha}x_1^{s_{i_l}}\right)$$

is a positive smooth function with a compact support in a small neighborhood of the origin. In the next chapter we shall prove that the maximal operator $\mathcal{M}_{ST}$ is bounded on $L^2(\mathbb{R}^3)$, adapting the ideas of [21].

4.3.4 Termination of the described procedure and remarks on the degenerate case

Recall that by Lemma 2.2.5 the homogeneous polynomial $\varphi_{\tilde{\kappa}^{(i_l)}}$ can be written in the form

$$\varphi_{\tilde{\kappa}^{(i_l)}}(x_1, x_2) = C_{A_{i_l},B_{i_l}}x_1^{A_{i_l}}x_2^{\tilde{B}_{i_l}}\prod_{r=1}^{L_{i_l}}(x_2^{q_{i_l}} - \lambda_r^{(i_l)}x_1^{p_{i_l}})^{n_r^{(i_l)}},$$

with $\lambda_r^{(i_l)} \in \mathbb{C} \setminus \{0\}$, L_{i_l}, $n_r^{(i_l)} \in \mathbb{N}$. Clearly, $\tilde{B}_{i_l} + q_{i_l}\sum_r n_r^{(i_l)} = B_{i_l} \geq 2$.
If we now assume that $\varphi_{\tilde{\kappa}^{(i_l)}}$ vanishes along the curve $\mathcal{C}_{I_l,\alpha}$, then the corresponding multiplicity $n_\alpha^{(i_l)}$ (or $\tilde{B}_{i_l}$) of the root is at least two, since $\partial_2\varphi_{\tilde{\kappa}^{(i_l)}}$ vanishes on the curve $\mathcal{C}_{I_l,\alpha}$ by definition. By Lemma 2.3.1 we conclude that in this case the Newton diagram $\mathcal{N}_d(\varphi_{\kappa^{(i_l)},\alpha})$ is exactly

$$[(q_{i_l}A_{i_l}, B_{i_l}), (A_\alpha, n_\alpha^{(i_l)})] \quad (\text{or } [(q_{i_l}A_{i_l}, B_{i_l}), (q_{i_l}\tilde{A}_{i_l}, \tilde{B}_{i_l})])$$

and

$$B_{i_l} \geq n_\alpha^{(i_l)} \geq 2.$$

Denote by $B_\alpha = n_\alpha^{(i_l)}$ (or $B_\alpha = \tilde{B}_{i_l}$). We see that the index α varies in some finite set and corresponds to some zero curve in $\mathcal{H}$ of multiplicity greater than or equal to two of the homogeneous polynomial $\varphi_{\tilde{\kappa}^{(i_l)}}$.
Lemma 2.3.1 also reveals that $B_{i_l} = B_\alpha$ if and only if $q_{i_l} = 1$ and

$$\varphi_{\tilde{\kappa}^{(i_l)}}(x_1, x_2) = C_{A_{i_l}, B_{i_l}} x_1^{A_{i_l}} \left(x_2 - z_{I_l,1} x_1^{p_{i_l}}\right)^{B_{i_l}}, \quad C_{A_{i_l}, B_{i_l}} \neq 0 \neq z_{I_l,1}.$$

Observe that it is possible that the described algorithm does not stop after a finite number of steps. Then there is a number L such that $B_{i_L} = B_{i_{L+j}}$ for every $j \in \mathbb{N}$. In this case we obtain a sequence of strictly increasing integers

$$p_L < p_{L+1} < p_{L+2} < \dots$$

tending to ∞ and

$$\varphi_{\tilde{\kappa}^{(i_{L+j})}}(x_1, x_2) = C_{A_{i_L}, B_{i_L}} x_1^{A_{i_L}} \left(x_2 - z_j x_1^{p_{L+j}}\right)^{B_{i_L}}.$$

In such a case we shall change variables

$$(x_1, x_2) \longmapsto \left(x_1, x_2 - r\left(x_1^{\frac{1}{Q_{I_L}}}\right)\right), \quad (x_1, x_2) \in \mathcal{H},$$

where $r(x_1) = \sum_{j=0}^{\infty} z_j x_1^{p_{L+j}}$. After this change of coordinates, we see that the new coordinates are of the form

$$\left(x_1, x_2 + z_{i_0,i_1} x_1^{s_{i_0}} + a(x_1), b\left(x_1^{\frac{1}{Q_{I_L}}}, x_2\right)\right),$$

where a is an analytic function in $x_1^{\frac{1}{N}}$ for some $N \in \mathbb{N}$, i.e. $a(x_1) = \tilde{a}(x_1^{\frac{1}{N}})$, and $\tilde{a}$ is a real-valued analytic function. Furthermore, we have $a(x_1) = o(x_1^{s_{i_0}})$ for $|x_1| \longrightarrow 0$. The function b is a real-valued analytic function. Its Newton polyhedron lies in the half-space $\{(t_1, t_2) : \ t_2 \geq B_{i_L}\}$ and contains the vertex (A_{i_L}, B_{i_L}). In particular, $\mathcal{T}(b)$ does not contain any point below the line $t_2 = B_{i_L}$. The coordinates in $\mathcal{H}$ are localized to a small neighborhood of the origin intersected the homogeneous domain

$$\left\{(x_1, x_2) \in \mathcal{H} : \; |x_2| \leq \delta x_1^{\frac{p_{i_L}}{Q_{I_L}}}\right\}.$$

The positive parameter δ can be chosen small enough. By the geometry of Newton polyhedron it is clear that the absolute value of the slope of any other edge of $\mathcal{N}_d(b)$, if at all existent, is strictly larger than $p_{i_L}^{-1}$. Since $B_{i_L} \geq 2$, the desired results follows if we apply again a bi-dyadic decomposition and after a usual rescaling argue exactly as in the first case.

5 Stopping time procedure

5.1 Preparation and general assumptions

In this chapter we shall prove the L^2-boundedness of the maximal operator

$$\mathcal{M}f(\cdot) = \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))\psi(x)dx, \quad f \geq 0. \tag{5.1}$$

The coordinates are parametrized by

$$\Phi(x_1, x_2) = \left(x_1, x_2 + \gamma(x_1^{\frac{1}{L}}), \phi(x_1^{\frac{1}{L}}, x_2)\right), \quad (x_1, x_2) \in \mathcal{H}.$$

The number L is a positive integer. The amplitude function ψ localizes the coordinates to a small homogeneous domain in $\mathcal{H}$ near the origin. More precisely, we have

$$\psi(x_1, x_2) = \eta\left(\frac{x_2}{\varepsilon x_1^{\frac{p}{L}}}\right)\tilde{\psi}(x_1, x_2), \quad (x_1, x_2) \in \mathcal{H},$$

where the function $\tilde{\psi}\colon \mathcal{H} \longrightarrow \mathbb{R}$ is a smooth positive bump function with $\operatorname{supp}\tilde{\psi} \subseteq U$, and U is a very small neighborhood of the origin in $\mathbb{R}^2$. The function η is the bump function defined in (4.8). Thus we integrate over

$$\operatorname{supp}\psi \cap \mathcal{H} \subseteq \left\{(x_1, x_2) \in \mathcal{H} : \ |x_2| \leq 2\varepsilon x_1^{\frac{p}{L}}\right\} \cap \operatorname{supp}\tilde{\psi}.$$

The number $\varepsilon = 2^{-M}$, $M \in \mathbb{N}$, is a sufficiently small dyadic parameter and p is a positive integer. Notice that due to the arguments in the previous chapter we are free to choose the neighborhood U and the parameter ε sufficiently small. Furthermore, ϕ is a real-valued analytic function with the properties

$$\phi(0,0)=0, \quad \nabla\phi(0,0)=(0,0).$$

Consider its Taylor expansion in a small neighborhood of the origin

$$\phi(x_1,x_2)=\sum_{\alpha,\beta=0}^{\infty} c_{\alpha,\beta}x_1^{\alpha}x_2^{\beta}.$$

The function ϕ can be decomposed in

$$\phi(x_1,x_2)=\phi_\kappa(x_1,x_2)+R_\kappa(x_1,x_2),$$

where ϕ_κ is the κ-homogeneous part of degree one of ϕ associated to the edge of $\mathcal{N}_d(\phi)$ lying on the line

$$L_\kappa=\{(t_1,t_2):\ \kappa_1 t_1+\kappa_2 t_2=1\}$$

with

$$\kappa=(\kappa_1,\kappa_2)=\left(\frac{1}{N},\frac{p}{N}\right), \quad N\in\mathbb{N}, \quad \gcd(p,N)=1.$$

The function R_κ is the analytic remainder term of ϕ consisting only of terms of higher κ-degree. Furthermore, from the previous chapter we can assume that

(i) $\gamma(x_1)=c_1x_1^n+r(x_1)$, where $c_1\in\mathbb{R}\setminus\{0\}$ and r is a real-valued analytic function with $r(x_1)=\mathcal{O}(x_1^{n+1})$;

(ii) $\phi_\kappa(x_1,x_2)=c_2x_1^N+x_2^B\left(c_3x_1^A+\ldots+c_4x_1^{\alpha}x_2^{\beta}\right)$ is a κ-homogeneous polynomial of degree one, $B\geq 2$, α, $\beta\in\mathbb{N}_0$, c_2, $c_3\in\mathbb{R}\setminus\{0\}$, $c_4\in\mathbb{R}$;

(iii) $N>\max\{n,L\}$, $n<p\leq\frac{N}{2}$, $L\neq n$.

The condition $p\leq\frac{N}{2}$ follows from the observation that the line L_κ contains the point (A,B), $B\geq 2$, and therefore $2\leq\frac{1}{\kappa_2}=\frac{N}{p}$. As already observed in [21], the main problem we have to deal with is the following. After the dyadic decomposition adapted to the weight $(L\kappa_1,\kappa_2)$ (cf. Lemma A.0.2) and usual rescaling arguments the considerations are reduced to the domain $x_1\sim 1$ and $|x_2|\ll 1$. If the integer B is very large and there are points $(s,t)\in\mathcal{T}(R_\kappa)$ below the line $t_2=B$, after the dyadic decomposition and rescaling such error terms will be of the form $2^{-jb(s,t)}c_{s,t}x_1^sx_2^t$, $b(s,t)>0$, $c_{s,t}\in\mathbb{R}\setminus\{0\}$. But since we localize to the domain where $|x_2|$ is very small, the perturbation terms $2^{-jb(s,t)}c_{s,t}x_1^sx_2^t$

can become much larger than $c_3x_1^Ax_2^B$, if $|x_2|$ is small enough, and have to be taken into account for the estimates of the corresponding oscillatory integral. We also refer the reader to Chapter 9 of the article [21].
We shall first describe the first step of the stopping time procedure and the $(l+1)$-th step, $l \geq 1$, in Section 5.3. At the the end of this chapter we shall prove that the described procedure terminates after finitely many steps.
Following [21] we decompose

$$\phi(x_1, x_2) = \phi(x_1, 0) + \theta(x_1, x_2).$$

Observe that $\phi(x_1, 0) = c_2x_1^N + \mathcal{O}(x_1^{N+1})$. Several cases may occur and each of those cases corresponds to certain geometric conditions on $\mathcal{N}(\theta)$. These geometric conditions are in turn connected to certain curvature conditions and relations between several dyadic parameters after an appropriate dyadic decomposition. Similar to the procedure described in the previous chapter each case requires a different approach. There will be one particular degenerate geometric condition, where the procedure does not stop and we will have to proceed to the next step. As in the previous chapter we need to make sure that the procedure will stop after finitely many steps.

5.2 Description of the first step of the stopping time procedure

First we assume that $\mathcal{N}(\theta)$ does not contain any point below the line $t_2 = B$.

5.2.1 Case 1: $\mathcal{N}(\theta) \subseteq \{(t_1, t_2): \ t_2 \geq B\}$

In this case we estimate the maximal operator by means of the bi-dyadic decomposition and the procedure stops at this stage. Using notations from the previous chapter, after the bi-dyadic decomposition and change of variables we obtain

$$\mathcal{M}f(\cdot) \leq \sum_{j,k=K}^{\infty} 2^{-j-k} \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi(2^{-j}x_1, 2^{-k}x_2)))\chi \otimes \chi(x)\psi(2^{-j}x_1, 2^{-k}x_2)dx.$$

The positive integer K can be assumed to be sufficiently large if the support of $\tilde{\psi}$ is small enough. We see that

$$\Phi(2^{-j}x_1, 2^{-k}x_2) = \begin{pmatrix} 2^{-j}x_1 \\ 2^{-k}x_2 + \gamma(2^{-\frac{j}{L}}x_1^{\frac{1}{L}}) \\ \phi(2^{-\frac{j}{L}}x_1^{\frac{j}{L}}, 0) + \theta(2^{-\frac{j}{L}}x_1^{\frac{j}{L}}, 2^{-k}x_2) \end{pmatrix}^T.$$

Several observations are in order. With arguments similar to those in the previous chapter we see that the sum is taken over pairs (j,k) with $2^{-k+j\frac{p}{L}} = 2^{-k+j\frac{\kappa_2}{\kappa_1 L}} \leq \sqrt{\varepsilon} \ll 1$. For the sake of completeness we give the precise argument. Since

$$\operatorname{supp} \chi \otimes \chi \subseteq \{x_1 : 2^{-1} \leq |x_1| \leq 2\} \times \{x_2 : 2^{-1} \leq |x_2| \leq 2\} \subseteq \mathbb{R}^2,$$

we conclude that for any $(x_1, x_2) \in \mathcal{H}$ with

$$\chi \otimes \chi(x)\psi(2^{-j}x_1, 2^{-k}x_2) = \chi \otimes \chi(x)\eta\left(\frac{2^{-k}x_2}{\varepsilon 2^{-\frac{jp}{L}}x_1^{\frac{p}{L}}}\right)\tilde{\psi}(2^{-j}x_1, 2^{-k}x_2) \neq 0$$

the estimate

$$\begin{aligned} 2^{-k-1} &\leq 2^{-k}|x_2| \\ &\leq 2\varepsilon 2^{-j\frac{p}{L}}x_1^{\frac{p}{L}} \\ &\leq 2^{1+\frac{p}{L}}\varepsilon 2^{-j\frac{p}{L}} \end{aligned}$$

holds true. This implies

$$2^{-k+j\frac{p}{L}} \leq 2^{2+\frac{p}{L}-M} \leq 2^{-\frac{M}{2}} = \sqrt{\varepsilon}$$

if M is chosen sufficiently large. Next, we see that

$$2^{-k}x_2 + \gamma(2^{-\frac{j}{L}}x_1^{\frac{1}{L}}) = 2^{-\frac{jn}{L}}\left(2^{-k+\frac{jn}{L}}x_2 + c_1 x_1^{\frac{n}{L}} + R_1^j(x_1)\right),$$

where R_1^j is a perturbation term. More precisely, the function R_1^j depends smoothly on the parameter $\delta_1 = 2^{-\frac{j}{L}}$ and is identically zero for $\delta_1 = 0$. By assumptions (i) and (iii) we have $n \neq L$ and $c_1 \neq 0$. Therefore for $(x_1, x_2) \in \operatorname{supp} \chi \otimes \chi$ we get

$$|\partial_1^2 c_1 x_1^{\frac{n}{L}}| \sim 1 \sim |\partial_1 c_1 x_1^{\frac{n}{L}}|.$$

From the assumption (iii) we conclude

$$2^{-k+\frac{jn}{L}} \leq 2^{-k+\frac{jp}{L}} \leq 2^{-\frac{M}{2}}.$$

Furthermore, we have

$$\begin{aligned}
\phi(2^{-\frac{j}{L}}x_1^{\frac{1}{L}}, 2^{-k}x_2) &= \phi(2^{-\frac{j}{L}}x_1^{\frac{1}{L}}, 0) + \theta(2^{-\frac{j}{L}}x_1^{\frac{1}{L}}, 2^{-k}x_2) \\
&= \phi(2^{-\frac{j}{L}}x_1^{\frac{1}{L}}, 0) + 2^{-j\frac{A}{L}-kB}\left(c_3 x_1^{\frac{A}{L}} x_2^B + R_3^{j,k}(x_1,x_2)\right) \\
&= 2^{-\frac{jN}{L}}\left(c_2 x_1^{\frac{N}{L}} + R_2^j(x_1) + 2^{j\frac{N-A}{L}-kB}\left(c_3 x_1^{\frac{A}{L}} x_2^B + R_3^{j,k}(x_1,x_2)\right)\right).
\end{aligned}$$

The function R_2^j is a perturbation term. As in Lemma 4.2.2 we also see that $R_3^{j,k}$ is a perturbation term, where we use the assumption that $\mathcal{N}(\theta) \subseteq \{(t_1,t_2) : \ t_2 \geq B\}$.
Since $(A,B) \in L_\kappa$, we get $N - A = pB$. We conclude

$$\begin{aligned}
2^{j\frac{N-A}{L}-kB} &= 2^{B(j\frac{N-A}{BL}-k)} \\
&= 2^{B(\frac{jp}{L}-k)} \\
&\leq 2^{-B\frac{M}{2}} \\
&\leq 2^{-\frac{M}{2}}.
\end{aligned}$$

If we apply Lemma 2.2.1 we see that it is sufficient to prove that each maximal operator

$$\mathcal{M}_{j,k}f(\cdot) = 2^{-j-k}\sup_{t>0}\int_{\mathbb{R}^2} f(\cdot - D_t^a(\Phi_{j,k}(x)))\chi\otimes\chi(x)\nu(x)dx, \ -k+j\frac{p}{L} \leq -\frac{M}{2}, \ \min\{j,k\} \geq K,$$

with

$$\Phi_{j,k}(x_1,x_2) = \begin{pmatrix} x_1 \\ 2^{-k+\frac{jn}{L}}x_2 + c_1 x_1^{\frac{n}{L}} + R_1^j(x_1) \\ c_2 x_1^{\frac{N}{L}} + R_2^j(x_1) + 2^{j\frac{N-A}{L}-kB} c_3 x_1^{\frac{A}{L}} x_2^B + 2^{j\frac{N-A}{L}-kB} R_3^{j,k}(x_1,x_2) \end{pmatrix}^T$$

is bounded on $L^2(\mathbb{R}^3)$ with the norm at most a constant multiple of $2^{-\delta(j+k)}$ for some $\delta > 0$.
The function ν is a smooth positive bump function supported on $\mathcal{H}$ and is identically one

on $\operatorname{supp}\chi\otimes\chi\cap\mathcal{H}$. Clearly, for $(x_1,x_2)\in\operatorname{supp}\chi\otimes\chi\cap\mathcal{H}$ we have

$$|\partial_2 c_3 x_1^{\frac{A}{L}} x_2^B| \sim 1 \sim |\partial_2^2 c_3 x_1^{\frac{A}{L}} x_2^B|,$$

since $B\geq 2$ and $c_3\neq 0$. The property (iii) gives

$$1\neq \frac{N}{L} > \frac{n}{L} \neq 1.$$

If we apply Corollary 3.3.10, we see that the corresponding oscillatory integral $\Lambda_{j,k}$ can be estimated by

$$|\Lambda_{j,k}(\lambda)|+|\nabla\Lambda_{j,k}(\lambda)| \lesssim 2^{-j-k}\min\left\{1, \frac{1}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}\min\{2^{-k+\frac{j}{nL}},(2^{j\frac{N-A}{L}-kB})^{\frac{1}{B}}\}}\right\}$$

if K and M are sufficiently large. Since $N>A$, we get

$$\frac{2^{-k}}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}\min\{2^{-k+\frac{j}{nL}},(2^{j\frac{N-A}{L}-kB})^{\frac{1}{B}}\}} \leq \frac{1}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}}.$$

Eventually, we conclude the uniform estimate

$$|\Lambda_{j,k}(\lambda)|+|\nabla\Lambda_{j,k}(\lambda)| \lesssim 2^{-j}\min\left\{2^{-k}, \frac{1}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}}\right\}.$$

Taking an appropriate geometric mean we obtain the desired result from Theorem 4.0.13, namely $\|\mathcal{M}_{j,k}\|_{L^2\to L^2}\lesssim 2^{-\delta(j+k)}$ for some $\delta>0$, which in turn implies the L^2-boundedness of $\mathcal{M}$ in (5.1). Thus the stopping time algorithm stops at this stage of the first step.

5.2.2 Case 2: $\mathcal{N}(\theta)\not\subseteq\{(t_1,t_2):\ t_2\geq B\}$

This means $\mathcal{N}_d(\theta)$ contains further vertices lying below the line $t_2=B$. Denote the closest vertex of $\mathcal{N}_d(\theta)$ which lies below the line $t_2=B$ by $(\tilde{A}_1,\tilde{B}_1)$. In particular, $1\leq\tilde{B}_1<B$. The edge $\mathfrak{E}_1=[(A,B),(\tilde{A}_1,\tilde{B}_1)]$ lies on the line

$$L_{\tilde{\kappa}^{(1)}}=\left\{(t_1,t_2):\ \tilde{\kappa}_1^{(1)}t_1+\tilde{\kappa}_2^{(1)}t_2=1\right\}$$

with the uniquely determined weight

$$\tilde{\kappa}^{(1)} = \left(\tilde{\kappa}_1^{(1)}, \tilde{\kappa}_2^{(1)}\right) = \left(\frac{q_1}{m_1}, \frac{p_1}{m_1}\right), \quad p_1, q_1, m_1 \in \mathbb{N}, \quad \gcd(p_1, q_1) = 1.$$

From the geometry of the Newton diagram we have

$$\frac{1}{p} = \frac{\kappa_1}{\kappa_2} > \frac{\tilde{\kappa}_1^{(1)}}{\tilde{\kappa}_2^{(1)}} = \frac{q_1}{p_1}. \tag{5.2}$$

Denote by $\theta_{\tilde{\kappa}^{(1)}}$ the $\tilde{\kappa}^{(1)}$-homogeneous part of θ of degree one which corresponds to the edge $\mathfrak{E}_1$, i.e.

$$\theta_{\tilde{\kappa}^{(1)}}(x_1, x_2) = \sum_{(\alpha,\beta)\in\mathfrak{E}_1\cap\mathbb{N}_0^2} c_{\alpha,\beta} x_1^\alpha x_2^\beta.$$

As in the previous algorithm we decompose the homogeneous domain $|x_2| \leq 2\varepsilon x_1^{\frac{p}{L}}$ into two domains

$$T_1 = \left\{(x_1, x_2) \in \mathcal{H} : \ N_1 x_1^{\frac{p_1}{q_1 L}} < |x_2| \leq 2\varepsilon x_1^{\frac{p}{L}}\right\},$$

called transition domain, and the homogeneous domain

$$H_1 = \left\{(x_1, x_2) \in \mathcal{H} : \ |x_2| \leq N_1 x_1^{\frac{p_1}{q_1 L}}\right\},$$

where $N_1 \in \mathbb{N}$. If the support of $\tilde{\psi}$ is chosen sufficiently small (or the neighborhood U of the origin is sufficiently small), then because of (5.2) the integer N_1 can be chosen sufficiently large and ε sufficiently small such that

$$H_1 \cap \operatorname{supp} \tilde{\psi} \subset \operatorname{supp} \psi.$$

As in the previous chapter these domains are treated in different ways. In the transition domain T_1 we stop our algorithm, since the situation can again be treated by means of the bi-dyadic decomposition. The arguments are the same as in the previous case and therefore omitted. The homogeneous domain H_1 requires further analysis. In order to localize the coordinates to the homogeneous domain H_1, set

$$\eta_{\tilde{\kappa}^{(1)}}(x_1, x_2) = \eta\left(\frac{x_2}{N_1 x_1^{\frac{p_1}{q_1 L}}}\right)\tilde{\psi}(x_1, x_2), \quad (x_1, x_2) \in \mathcal{H}.$$

It is enough to show that the maximal operator

$$\mathcal{M}_{\tilde{\kappa}^{(1)}} f(\cdot) = \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))\eta_{\tilde{\kappa}^{(1)}}(x)dx \tag{5.3}$$

is bounded on $L^2(\mathbb{R}^3)$. Several different cases may occur and these cases will depend on the position of the point $(N-n,1)$ with respect to the line $L_{\tilde{\kappa}^{(1)}}$. As we shall see the procedure does not stop in the first step if and only if the point $(N-n,1)$ lies above the line $L_{\tilde{\kappa}^{(1)}}$ and in addition $\partial_2\theta_{\tilde{\kappa}^{(1)}}$ has zeros in $\mathcal{H}$.

Case 2.1: $\tilde{\kappa}_1(N-n) + \tilde{\kappa}_2 \leq 1$

This case means that the point $(N-n,1)$ lies below or on the line $L_{\tilde{\kappa}^{(1)}}$. Similar to the previous chapter we decompose dyadically with respect to the weight $(L\tilde{\kappa}_1^{(1)}, \tilde{\kappa}_2^{(1)})$. More precisely, as described in Lemma A.0.2 we can write

$$\sum_{j=K}^{\infty} \rho_j(x) = 1 \quad \text{for } x \in \operatorname{supp}\tilde{\psi} \setminus \{0\},$$

where we set

$$\rho_j(x) = \rho(\delta_{2^j}(x)) \quad \text{and} \quad \delta_r(x_1,x_2) = (r^{\tilde{\kappa}_1^{(1)}L}x_1, r^{\tilde{\kappa}_2^{(1)}}x_2).$$

The positive integer K can be assumed to be sufficiently large if $\operatorname{supp}\tilde{\psi}$ is small enough. We get

$$\begin{aligned}
\mathcal{M}_{\tilde{\kappa}^{(1)}} f(\cdot) &\leq \sum_{j=K}^{\infty} \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))\rho_j(x)\eta_{\tilde{\kappa}^{(1)}}(x)dx \\
&= \sum_{j=K}^{\infty} 2^{-j(L\tilde{\kappa}_1^{(1)}+\tilde{\kappa}_2^{(1)})} \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi(\delta_{2^{-j}}(x))))\rho(x)\eta_{\tilde{\kappa}^{(1)}}(\delta_{2^{-j}}(x))dx,
\end{aligned}$$

where, in the last step, we changed variables. First observe that Lemma A.0.2 implies that in $\mathcal{H}$ the function

$$x \longmapsto \rho(x)\eta\left(\frac{x_2}{N_1 x_1^{\frac{p_1}{q_1}L}}\right)$$

is supported in

$$\left\{(x_1,x_2)\in\mathcal{H}:\frac{1}{\max\left\{5,(10N_1)^{\frac{q_1L}{p_1}}\right\}}\leq x_1\leq R,\,|x_2|\leq R\right\},\tag{5.4}$$

where $R=3\cdot 2^{\max\{L\tilde{\kappa}_1^{(1)},\tilde{\kappa}_2^{(1)}\}}$. In particular, $x_1\sim 1$ and $|x_2|\lesssim 1$. We remark that, although N_1 is assumed to be large, we are still free to choose U (or $\operatorname{supp}\tilde{\psi}$) sufficiently small with respect to N_1. Furthermore, we see that

$$\Phi(\delta_{2^{-j}}(x))=\begin{pmatrix}2^{-j\tilde{\kappa}_1^{(1)}L}x_1\\ 2^{-j\tilde{\kappa}_1^{(1)}n}\left(2^{-j(\tilde{\kappa}_2^{(1)}-\tilde{\kappa}_1^{(1)}n)}x_2+c_1x_1^{\frac{n}{L}}+R_1^j(x_1)\right)\\ 2^{-j\tilde{\kappa}_1^{(1)}N}\left(c_2x_1^{\frac{N}{L}}+R_2^j(x_1)+2^{-j(1-\tilde{\kappa}_1^{(1)}N)}\left(\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}},x_2)+R_3^j(x_1,x_2)\right)\right)\end{pmatrix}^T.$$

The functions R_1^j, R_2^j and R_3^j are smooth perturbations. After applying Lemma 2.2.1 we see that it is enough to show that every maximal operator

$$\mathcal{M}^j_{\tilde{\kappa}^{(1)}}f(\cdot)=2^{-j(L\tilde{\kappa}_1^{(1)}+\tilde{\kappa}_2^{(1)})}\sup_{t>0}\int_{\mathbb{R}^2}f(\cdot-D_t^a(\Phi_j(x)))\mu(x)\rho(x)\eta_{\tilde{\kappa}^{(1)}}(\delta_{2^{-j}}(x))dx,\quad j\geq K,$$

with

$$\Phi_j(x)=\begin{pmatrix}x_1\\ 2^{-j(\tilde{\kappa}_2^{(1)}-\tilde{\kappa}_1^{(1)}n)}x_2+c_1x_1^{\frac{n}{L}}+R_1^j(x_1)\\ c_2x_1^{\frac{N}{L}}+R_2^j(x_1)+2^{-j(1-\tilde{\kappa}_1^{(1)}N)}\left(\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}},x_2)+R_3^j(x_1,x_2)\right)\end{pmatrix}^T,$$

is bounded on $L^2(\mathbb{R}^3)$ with the norm at most a constant multiple of $2^{-j\delta}$ for some $\delta>0$. The function μ is a smooth positive bump function, supported in $\mathcal{H}$ and identically one on the set in (5.4). First notice that $\frac{1}{\tilde{\kappa}_1^{(1)}}>N$ and therefore $2^{-j(1-\tilde{\kappa}_1^{(1)}N)}\ll 1$. Using the assumption (iii) we get $\tilde{\kappa}_2^{(1)}>\tilde{\kappa}_1^{(1)}p>\tilde{\kappa}_1^{(1)}n$, and therefore $2^{-j(\tilde{\kappa}_2^{(1)}-n\tilde{\kappa}_1^{(1)})}\ll 1$.
We also have the identity

$$\frac{2^{-j(1-\tilde{\kappa}_1^{(1)}N)}}{2^{-j(\tilde{\kappa}_2^{(1)}-n\tilde{\kappa}_1^{(1)})}}=2^{j(\tilde{\kappa}_1^{(1)}(N-n)+\tilde{\kappa}_2^{(1)}-1)}.$$

This term is either equal to one in the case that $(N-n,1)$ lies on the line $L_{\tilde\kappa^{(1)}}$ or very small if this point lies strictly below $L_{\tilde\kappa^{(1)}}$. We investigate these cases. Observe that, in any case, for $x_1 \sim 1$ we have $|\partial_2^B \theta_{\tilde\kappa^{(1)}}(x_1^{\frac{1}{L}}, x_2)| \sim 1$.

Case 2.1.1: $\tilde\kappa_1(N-n)+\tilde\kappa_2 < 1$

In this case we have

$$\frac{2^{-j(1-\tilde\kappa_1^{(1)}N)}}{2^{-j(\tilde\kappa_2^{(1)}-n\tilde\kappa_1^{(1)})}} \ll 1.$$

Therefore, if we apply Lemma 3.3.7, we see that the oscillatory integral $\Lambda^j_{\tilde\kappa^{(1)}}$ associated to the maximal operator $\mathcal{M}^j_{\tilde\kappa^{(1)}}$ can be estimated by

$$\left|\nabla\Lambda^j_{\tilde\kappa^{(1)}}(\lambda)\right| + \left|\Lambda^j_{\tilde\kappa^{(1)}}(\lambda)\right| \lesssim \frac{2^{-j(L\tilde\kappa_1^{(1)}+\tilde\kappa_2^{(1)})}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}} \min\left\{2^{-j\frac{1-\tilde\kappa_1^{(1)}N}{B}}, 2^{-j(\tilde\kappa_2^{(1)}-\tilde\kappa_1^{(1)}n)}\right\}},$$

if K is sufficiently large. Using the identity

$$\frac{1}{\min\{\frac{1}{A},\frac{1}{B}\}} = \max\{A,B\} \quad \text{for any } A, B \in \mathbb{R}_{>0},$$

we conclude

$$\left|\nabla\Lambda^j_{\tilde\kappa^{(1)}}(\lambda)\right| + \left|\Lambda^j_{\tilde\kappa^{(1)}}(\lambda)\right| \lesssim 2^{-j(L\tilde\kappa_1^{(1)}+\tilde\kappa_2^{(1)})} \cdot \frac{\max\{2^{j\frac{1-\tilde\kappa_1^{(1)}N}{B}}, 2^{j(\tilde\kappa_2^{(1)}-\tilde\kappa_1^{(1)}n)}\}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}}}.$$

Observe that the estimate

$$\begin{aligned}
\frac{1-\tilde\kappa_1^{(1)}N}{B} &= \tilde\kappa_1^{(1)} \cdot \frac{\frac{1}{\tilde\kappa_1^{(1)}} - N}{B} \\
&< \tilde\kappa_1^{(1)} \cdot \frac{\frac{1}{\tilde\kappa_1^{(1)}} - A}{B} \\
&= \tilde\kappa_1^{(1)} \cdot \frac{\tilde\kappa_2^{(1)}}{\tilde\kappa_1^{(1)}} \\
&= \tilde\kappa_2^{(1)}
\end{aligned}$$

holds true. The geometric interpretation of this estimate is that the point (N, B) lies above the line $L_{\tilde{\kappa}^{(1)}}$. We conclude

$$\left|\nabla \Lambda^j_{\tilde{\kappa}^{(1)}}(\lambda)\right| + \left|\Lambda^j_{\tilde{\kappa}^{(1)}}(\lambda)\right| \lesssim \frac{2^{-jL\tilde{\kappa}_1^{(1)}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}}}.$$

Theorem 4.0.13 implies that $\|\mathcal{M}^j_{\tilde{\kappa}^{(1)}}\|_{L^2\to L^2} \lesssim 2^{-jL\tilde{\kappa}_1^{(1)}}$. Eventually, Minkowski's inequality implies that the maximal operator $\mathcal{M}_{\tilde{\kappa}^{(1)}}$ in (5.3) is bounded on $L^2(\mathbb{R}^3)$.

Case 2.1.2: $\tilde{\kappa}_1(N-n) + \tilde{\kappa}_2 = 1$

In particular, we have

$$2^{-j(1-\tilde{\kappa}_1^{(1)}N)} = 2^{-j(\tilde{\kappa}_2^{(1)} - n\tilde{\kappa}_1^{(1)})}.$$

The arguments will now depend on whether or not we localize near the zeros of $\partial_2\theta_{\tilde{\kappa}^{(1)}}(\cdot^{\frac{1}{L}}, \cdot)$. The zeros of $\partial_2\theta_{\tilde{\kappa}^{(1)}}$ in $\mathcal{H}$ (if at all existent) are finitely many curves

$$\mathcal{C}_i = \left\{(x_1, z_i x_1^{\frac{p_1}{q_1}}) : \ x_1 > 0\right\}, \quad z_i \in \mathbb{R}.$$

The index i varies in some finite range which we shall not specify here. In order to localize near the zeros of $\partial_2\theta_{\tilde{\kappa}^{(1)}}(\cdot^{\frac{1}{L}}, \cdot)$, we set

$$\eta^i_{\tilde{\kappa}^{(1)}}(x_1, x_2) = \eta\left(\frac{x_2 - z_i x_1^{\frac{p_1}{q_1 L}}}{\varepsilon_i x_1^{\frac{p_1}{q_1 L}}}\right), \quad (x_1, x_2) \in \mathcal{H},$$

with $\varepsilon_i > 0$ sufficiently small. Notice that every $\eta^i_{\tilde{\kappa}^{(1)}}$ is $(L\tilde{\kappa}_1^{(1)}, \tilde{\kappa}_2^{(1)})$-homogeneous of degree zero and that $\operatorname{supp}\eta^i_{\tilde{\kappa}^{(1)}} \cap \mathcal{H}$ are disjoint on $\operatorname{supp}\rho$, if ε_i are small enough. We get

$$\mathcal{M}^j_{\tilde{\kappa}^{(1)}} f \le \overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}} f + \sum_i \mathcal{M}^j_{\tilde{\kappa}^{(1)},i} f, \quad j \ge K,$$

where

$$\mathcal{M}^j_{\tilde{\kappa}^{(1)},i} f(\cdot) = 2^{-j(L\tilde{\kappa}_1^{(1)} + \tilde{\kappa}_2^{(1)})} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi_j(x)))\mu(x)\rho(x)\eta^i_{\tilde{\kappa}^{(1)}}(x)\eta_{\tilde{\kappa}^{(1)}}(\delta_{2^{-j}}(x))dx,$$

$$\overline{\mathcal{M}}^j_{\tilde\kappa^{(1)}} f(\cdot) = 2^{-j(L\tilde\kappa_1^{(1)}+\tilde\kappa_2^{(1)})} \sup_{t>0} \int\limits_{\mathbb{R}^2} f(\cdot - D_t^a(\Phi_j(x)))\mu(x)\rho(x)(1-\sum_i \eta^i_{\tilde\kappa^{(1)}}(x))\eta_{\tilde\kappa^{(1)}}(\delta_{2^{-j}}(x))dx.$$

First, we discuss the L^2-boundedness of the maximal operators $\mathcal{M}^j_{\tilde\kappa^{(1)},i}$. Since for every i and every $x_1 > 0$ we have $\partial_2\theta_{\tilde\kappa^{(1)}}(x_1^{\frac{1}{L}}, z_i x_1^{\frac{p_1}{q_1 L}}) = 0$, we obtain from Lemma 3.3.7 that the corresponding oscillatory integrals $\Lambda^j_{\tilde\kappa^{(1)},i}$ can be estimated by

$$\left|\nabla\Lambda^j_{\tilde\kappa^{(1)},i}(\lambda)\right| + \left|\Lambda^j_{\tilde\kappa^{(1)},i}(\lambda)\right| \lesssim \frac{2^{-j(L\tilde\kappa_1^{(1)}+\tilde\kappa_2^{(1)})}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}} \min\left\{2^{-j\frac{1-\tilde\kappa_1^{(1)}N}{B}}, 2^{-j(\tilde\kappa_2^{(1)}-\tilde\kappa_1^{(1)}n)}\right\}}$$

if the parameters ε_i are sufficiently small and the integer K is sufficiently large. This implies

$$\begin{aligned}\left|\nabla\Lambda^j_{\tilde\kappa^{(1)},i}(\lambda)\right| + \left|\Lambda^j_{\tilde\kappa^{(1)},i}(\lambda)\right| &\lesssim \frac{2^{-j(L\tilde\kappa_1^{(1)}+\tilde\kappa_2^{(1)})}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}} 2^{-j(\tilde\kappa_2^{(1)}-\tilde\kappa_1^{(1)}n)}} \\ &\lesssim \frac{2^{-jL\tilde\kappa_1^{(1)}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}}}.\end{aligned}$$

From Theorem 4.0.13 we conclude that $\|\mathcal{M}^j_{\tilde\kappa^{(1)},i}\|_{L^2\to L^2} \lesssim 2^{-jL\tilde\kappa_1^{(1)}}$.
Next, we turn our attention to $\overline{\mathcal{M}}^j_{\tilde\kappa^{(1)}}$. Here we shall again distinguish between two different cases. In fact, the estimate of the oscillatory integral will depend on the fact whether or not $\partial_2^2\theta_{\tilde\kappa^{(1)}}(\cdot^{\frac{1}{L}}, \cdot)$ vanishes. Since $\partial_2^2\theta_{\tilde\kappa^{(1)}}$ is also $\tilde\kappa^{(1)}$-homogeneous and $\tilde\kappa_2^{(1)} \leq \frac{1}{2} < 1$, we conclude that the zeros of $\partial_2^2\theta_{\tilde\kappa^{(1)}}$ in $\mathcal{H}$ (if at all existent) are finitely many curves

$$\overline{\mathcal{C}}_\alpha = \left\{(x_1, \overline{z}_\alpha x_1^{\frac{p_1}{q_1}}) : \; x_1 > 0\right\}, \quad \overline{z}_\alpha \in \mathbb{R},$$

where the index α varies in some finite range. We localize again to some narrow homogeneous domains near the zeros of $\partial_2^2\theta_{\tilde\kappa^{(1)}}(\cdot^{\frac{1}{L}}, \cdot)$ (and away from the zeros of $\partial_2\theta_{\tilde\kappa^{(1)}}(\cdot^{\frac{1}{L}}, \cdot)$) by

$$\overline{\eta}^\alpha_{\tilde\kappa^{(1)}}(x_1, x_2) = \eta\left(\frac{x_2 - \overline{z}_\alpha x_1^{\frac{p_1}{q_1 L}}}{\overline{\varepsilon}_\alpha x_1^{\frac{p_1}{q_1 L}}}\right)\mu(x)\rho(x)\left(1 - \sum_i \eta^i_{\tilde\kappa^{(1)}}(x)\right),$$

with $\overline{\varepsilon}_\alpha > 0$ sufficiently small. Set

$$\overline{\overline{\eta}}_{\tilde{\kappa}^{(1)}}(x_1,x_2) = \left(1 - \sum_\alpha \eta\left(\frac{x_2 - \overline{z}_\alpha x_1^{\frac{p_1}{q_1 L}}}{\overline{\varepsilon}_\alpha x_1^{\frac{p_1}{q_1 L}}}\right)\right)\mu(x)\rho(x)\left(1 - \sum_i \eta^i_{\tilde{\kappa}^{(1)}}(x)\right).$$

We get

$$\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}} f \leq \sum_\alpha \overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)},\alpha} f + \overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(1)}} f,$$

with

$$\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)},\alpha} f(\cdot) = 2^{-j(L\tilde{\kappa}_1^{(1)} + \tilde{\kappa}_2^{(1)})} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi_j(x)))\overline{\eta}^\alpha_{\tilde{\kappa}^{(1)}}(x)\eta_{\tilde{\kappa}^{(1)}}(\delta_{2^{-j}}(x))dx, \quad j \geq K,$$

$$\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(1)}} f(\cdot) = 2^{-j(L\tilde{\kappa}_1^{(1)} + \tilde{\kappa}_2^{(1)})} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi_j(x)))\overline{\overline{\eta}}_{\tilde{\kappa}^{(1)}}(x)\eta_{\tilde{\kappa}^{(1)}}(\delta_{2^{-j}}(x))dx, \quad j \geq K.$$

It is sufficient to show that

$$\|\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(1)}}\|_{L^2\to L^2} + \max_\alpha \|\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)},\alpha}\|_{L^2\to L^2} \lesssim 2^{-jL\tilde{\kappa}_1^{(1)}}. \tag{5.5}$$

First, we analyze $\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(1)}}$. Notice that for every $x \in \operatorname{supp} \overline{\overline{\eta}}_{\tilde{\kappa}^{(1)}}$ we have

$$\partial_2\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, x_2) \neq 0, \quad \partial_2^2\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, x_2) \neq 0.$$

Therefore, by Corollary 3.3.10 the corresponding oscillatory integral $\overline{\overline{\Lambda}}^j_{\tilde{\kappa}^{(1)}}$ can be estimated by

$$\begin{aligned}
\left|\nabla\overline{\overline{\Lambda}}^j_{\tilde{\kappa}^{(1)}}(\lambda)\right| + \left|\overline{\overline{\Lambda}}^j_{\tilde{\kappa}^{(1)}}(\lambda)\right| &\lesssim 2^{-j(L\tilde{\kappa}_1^{(1)} + \tilde{\kappa}_2^{(1)})} \cdot \frac{(1+|\lambda|)^{-\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}}{\min\left\{2^{-j\frac{1-\tilde{\kappa}_1^{(1)}N}{B}}, 2^{-j(\tilde{\kappa}_2^{(1)} - \tilde{\kappa}_1^{(1)}n)}\right\}} \\
&\lesssim 2^{-j(L\tilde{\kappa}_1^{(1)} + \tilde{\kappa}_2^{(1)})} \cdot \frac{2^{j\tilde{\kappa}_2^{(1)}}}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}} \\
&\lesssim \frac{2^{-jL\tilde{\kappa}_1^{(1)}}}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B},\frac{5}{6}\}}}
\end{aligned}$$

if K is sufficiently large. We obtain $\|\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(1)}}\|_{L^2\to L^2} \lesssim 2^{-jL\tilde{\kappa}_1^{(1)}}$.
In order to estimate the maximal operators $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)},\alpha}$, we first observe that Lemma 2.2.4 implies that if for some α we have $\overline{z}_\alpha \neq z_i$ for every i, then

$$\partial_2\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, \overline{z}_\alpha x_1^{\frac{p_1}{q_1 L}}) \neq 0, \quad \partial_1\partial_2\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, \overline{z}_\alpha x_1^{\frac{p_1}{q_1 L}}) \neq 0, \quad \partial_2^2\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, \overline{z}_\alpha x_1^{\frac{p_1}{q_1 L}}) = 0$$

hold true for every $x_1 > 0$. Recall that $|\partial_2^B\theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, x_2)| \sim 1$ on $\operatorname{supp} \overline{\eta}^\alpha_{\tilde{\kappa}^{(1)}}$. Lemma 3.3.12 implies that the corresponding oscillatory integrals $\overline{\Lambda}^j_{\tilde{\kappa}^{(1)},\alpha}$ can be estimated by

$$\left|\nabla\overline{\Lambda}^j_{\tilde{\kappa}^{(1)},\alpha}(\lambda)\right| + \left|\overline{\Lambda}^j_{\tilde{\kappa}^{(1)},\alpha}(\lambda)\right| \lesssim 2^{-j(L\tilde{\kappa}_1^{(1)}+\tilde{\kappa}_2^{(1)})} \cdot \frac{1}{(1+|\lambda|)^{\frac{1}{2}+b}2^{-j(\tilde{\kappa}_2^{(1)}-\tilde{\kappa}_1^{(1)}n)}},$$

if K is sufficiently large and the numbers ε_i, $\overline{\varepsilon}_\alpha$ and b are sufficiently small. This gives

$$\left|\nabla\overline{\Lambda}^j_{\tilde{\kappa}^{(1)},\alpha}(\lambda)\right| + \left|\overline{\Lambda}^j_{\tilde{\kappa}^{(1)},\alpha}(\lambda)\right| \lesssim \frac{2^{-jL\tilde{\kappa}_1^{(1)}}}{(1+|\lambda|)^{\frac{1}{2}+b}}.$$

In particular, $\|\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)},\alpha}\|_{L^2\to L^2} \lesssim 2^{-jL\tilde{\kappa}_1^{(1)}}$. This implies that the inequality (5.5) is true. Thus we are left with the last case, where the point $(N-n,1)$ lies above the line $L_{\tilde{\kappa}^{(1)}}$.

Case 2.2: $\tilde{\kappa}_1^{(1)}(N-n)+\tilde{\kappa}_2^{(1)} > 1$

We proceed as follows. First, observe that we get the pointwise estimate

$$\begin{aligned}\mathcal{M}_{\tilde{\kappa}^{(1)}}f(\cdot) \quad &\leq \quad \sum_i \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))\eta^i_{\tilde{\kappa}^{(1)}}(x)\tilde{\psi}(x)dx \\ &\quad + \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))(1-\sum_i \eta^i_{\tilde{\kappa}^{(1)}}(x))\eta_{\tilde{\kappa}^{(1)}}(x)dx.\end{aligned}$$

Set

$$\mathcal{M}_{\tilde{\kappa}^{(1)},i}f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi(x)))\eta^i_{\tilde{\kappa}^{(1)}}(x)\tilde{\psi}(x)dx.$$

If we decompose the second maximal operator dyadically with respect to the weight $(L\tilde{\kappa}_1^{(1)}, \tilde{\kappa}_2^{(1)})$ and apply the usual rescaling argument, we see that we are left with the sum of maximal operators

$$\sum_i \mathcal{M}_{\tilde{\kappa}^{(1)},i} + \sum_{j=K}^{\infty} \overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}},$$

where $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}}$ is the maximal operator from the previous case given by

$$\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}} f(\cdot) = 2^{-j(L\tilde{\kappa}_1^{(1)}+\tilde{\kappa}_2^{(1)})} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D_t^a(\Phi_j(x)))\mu(x)\rho(x)(1-\sum_i \eta^i_{\tilde{\kappa}^{(1)}}(x))\eta_{\tilde{\kappa}^{(1)}}(\delta_{2^{-j}}(x))dx.$$

Notice that for every $x \in \operatorname{supp} \mu\rho(1-\sum_i \eta^i_{\tilde{\kappa}^{(1)}})$ we have $\partial_2 \theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, x_2) \neq 0$ and that the ratio

$$\frac{2^{-j(1-\tilde{\kappa}_1^{(1)}N)}}{2^{-j(\tilde{\kappa}_2^{(1)}-n\tilde{\kappa}_1^{(1)})}} = 2^{j(\tilde{\kappa}_1^{(1)}(N-n)+\tilde{\kappa}_2^{(1)}-1)}$$

converges to infinity for $j \longrightarrow \infty$. For every $x \in \operatorname{supp} \mu\rho(1-\sum_i \eta^i_{\tilde{\kappa}^{(1)}})$ we also have $\partial_2^B \theta_{\tilde{\kappa}^{(1)}}(x_1^{\frac{1}{L}}, x_2) \neq 0$. Thus Lemma 3.3.11 implies that the oscillatory integrals $\overline{\Lambda}^j_{\tilde{\kappa}^{(1)}}$ corresponding to the maximal operators $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}}$ can be estimated by

$$\begin{aligned} \left|\nabla \overline{\Lambda}^j_{\tilde{\kappa}^{(1)}}(\lambda)\right| + \left|\overline{\Lambda}^j_{\tilde{\kappa}^{(1)}}(\lambda)\right| &\lesssim 2^{-j(L\tilde{\kappa}_1^{(1)}+\tilde{\kappa}_2^{(1)})} \cdot \frac{(1+|\lambda|)^{-\frac{1}{2}-\frac{1}{B}}}{\min\left\{2^{-j\frac{1-\tilde{\kappa}_1^{(1)}N}{B}}, 2^{-j(\tilde{\kappa}_2^{(1)}-\tilde{\kappa}_1^{(1)}n)}\right\}} \\ &\lesssim 2^{-j(L\tilde{\kappa}_1^{(1)}+\tilde{\kappa}_2^{(1)})} \cdot \frac{2^{j\tilde{\kappa}_2^{(1)}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}}} \\ &\lesssim \frac{2^{-jL\tilde{\kappa}_1^{(1)}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B}}}, \end{aligned}$$

if the integer K is large enough. Theorem 4.0.13, together with Minkowski's inequality, shows that the sum $\sum_{j=K}^{\infty} \overline{\mathcal{M}}^j_{\tilde{\kappa}^{(1)}}$ is bounded on $L^2(\mathbb{R}^3)$.

We are left with the finite sum of maximal operators $\mathcal{M}_{\tilde{\kappa}^{(1)},i}$. Similar to the procedure in the previous chapter we advance to the next step changing variables

$$\zeta_i(x_1, x_2) = \left(x_1, x_2 - z_i x_1^{\frac{p_1}{q_1 L}}\right), \quad (x_1, x_2) \in \mathcal{H}.$$

Then

$$\Phi_i(x) = \Phi(\zeta_i^{-1}(x_1, x_2)) = \begin{pmatrix} x_1 \\ x_2 + \gamma(x_1^{\frac{1}{L}}) + z_i x_1^{\frac{p_1}{q_1 L}} \\ \phi_i(x_1^{\frac{1}{q_1 L}}, x_2) \end{pmatrix}^T, \quad (x_1, x_2) \in \mathcal{H}, \tag{5.6}$$

where

$$\begin{aligned} \phi_i(x_1^{\frac{1}{q_1 L}}, x_2) &= \phi\left(x_1^{\frac{1}{L}}, x_2 + z_i x_1^{\frac{p_1}{q_1 L}}\right) \\ &= c_2 x_1^{\frac{N}{L}} + o(x_1^{\frac{N}{L}}) + \theta(x_1^{\frac{1}{L}}, x_2 + z_i x_1^{\frac{p_1}{q_1 L}}). \end{aligned}$$

As in Lemma 2.3.1 we write

$$\theta(x_1^{\frac{1}{L}}, x_2 + z_i x_1^{\frac{p_1}{q_1 L}}) = \tilde{\theta}_i(x_1^{\frac{1}{q_1 L}}, x_2).$$

The function $\tilde{\theta}_i$ is real-valued and analytic. Clearly, it is highly possible that

$$\mathcal{T}(\tilde{\theta}_i) \cap (\mathbb{R} \times \{0\}) \neq \emptyset.$$

On the other hand, if we decompose

$$\tilde{\theta}_i(x_1, x_2) = \tilde{\theta}_i(x_1, 0) + \theta_i(x_1, x_2),$$

then clearly, $\mathcal{T}(\theta_i) \subseteq \{(t_1, t_2) : \ t_2 \geq 1\}$. Several observations are in order. First, notice that $\tilde{\theta}_i(x_1, 0) = \mathcal{O}(x_1^{m_1})$ and $m_1 > q_1 N$, since $N < \frac{1}{\tilde{\kappa}_1^{(1)}} = \frac{m_1}{q_1}$. The Newton diagram $\mathcal{N}_d(\theta_i)$ contains the face $[(q_1 A, B), (A_i, B_i)]$, $B \geq B_i$, which possibly even degenerates to a vertex, lying on the line

$$L_{\kappa^{(1)}} = \left\{(t_1, t_2) : \kappa_1^{(1)} t_1 + \kappa_2^{(1)} t_2 = 1\right\},$$

where

$$\kappa^{(1)} = (\kappa_1^{(1)}, \kappa_2^{(1)}) = \left(\frac{\tilde{\kappa}_1^{(1)}}{q_1}, \tilde{\kappa}_2^{(1)}\right) = \left(\frac{1}{m_1}, \frac{p_1}{m_1}\right).$$

Example 5.2.1. Let k be a large positive even integer. Consider

$$\theta(x_1, x_2) = (x_2 - x_1^2)^k - x_1^{2k}.$$

We have $\mathcal{T}(\theta) \subseteq \{(t_1, t_2) : \ t_2 \geq 1\}$. Then $\partial_2 \theta$ vanishes along the parabola (x_1, x_1^2). If we change coordinates $y_1 = x_1$, $y_2 = x_2 - x_1^2$, then in the new coordinates θ is given by

$\tilde{\theta}(y_1, y_2) = y_2^k - y_1^{2k}$. This gives $\mathcal{T}(\tilde{\theta}) = \{(2k, 0), (0, k)\}$.

If we denote by $\theta_{\kappa^{(1)},i}$ the $\kappa^{(1)}$-homogeneous part of θ_i of degree one which corresponds to the face $[(q_1A, B), (A_i, B_i)]$ of $\mathcal{N}_d(\theta_i)$, i.e. $\mathcal{T}(\theta_{\kappa^{(1)},i}) \subseteq [(q_1A, B), (A_i, B_i)]$, we can decompose θ_i as usual

$$\theta_i(x_1, x_2) = \theta_{\kappa^{(1)},i}(x_1, x_2) + R_{\kappa^{(1)},i}(x_1, x_2),$$

where $R_{\kappa^{(1)},i}$ is the analytic remainder term of higher $\kappa^{(1)}$-degree. Eventually, this gives

$$\phi_i(x_1^{\frac{1}{q_1L}}, x_2) = c_2 x_1^{\frac{N}{L}} + o(x_1^{\frac{N}{L}}) + \theta_{\kappa^{(1)},i}(x_1^{\frac{1}{q_1L}}, x_2) + R_{\kappa^{(1)},i}(x_1^{\frac{1}{q_1L}}, x_2).$$

Lemma 2.3.1 also shows that $B_i \geq 2$. Also notice that since $\frac{p_1}{q_1} > p > n$, we can conclude that

$$\gamma(x_1^{\frac{1}{L}}) + z_i x_1^{\frac{p_1}{q_1L}} = c_1 x_1^{\frac{n}{L}} + r_i(x_1^{\frac{1}{L}}),$$

where r_i is an analytic function in some fractional power with $r_i(x_1) = o(x_1^n)$.
Observe that since for $(x_1, x_2) \in \mathcal{H}$

$$\eta^i_{\tilde{\kappa}^{(1)}}(x_1, x_2 + z_i x_1^{\frac{p_1}{q_1L}})\tilde{\psi}(x_1, x_2 + z_1 x_1^{\frac{p_1}{q_1L}}) \quad = \quad \eta\left(\frac{x_2}{\varepsilon_i x_1^{\frac{p_1}{q_1L}}}\right)\underbrace{\tilde{\psi}(x_1, x_2 + z_1 x_1^{\frac{p_1}{q_1L}})}_{\psi_i(x_1,x_2)},$$

we see that in $\mathcal{H}$ the coordinates are now localized to the homogeneous domain

$$\left\{(x_1, x_2) \in \mathcal{H} : \; |x_2| \leq 2\varepsilon_i x_1^{\frac{p_1}{q_1L}}\right\}$$

in a small neighborhood near the origin in $\mathcal{H}$. The support of ψ_i is small if the support of $\tilde{\psi}$ is small. We can assume $\varepsilon_i = 2^{-M_i}$, $M_i \in \mathbb{N}$, to be a sufficiently small dyadic number. In the end, we have to show that every maximal operator

$$\mathcal{M}_i f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_i(x)))\eta_i(x)dx, \quad f \geq 0,$$

where

$$\eta_i(x_1, x_2) = \eta\left(\frac{x_2}{\varepsilon_i x_1^{\frac{p_1}{q_1L}}}\right)\psi_i(x_1, x_2),$$

and Φ_i as given in (5.6), is bounded on $L^2(\mathbb{R}^3)$. We remark that the procedure does not stop in the first step, because $(N-n,1)$ lies above the line $L_{\tilde{\kappa}^{(1)}}$ or, equivalently, the point $(q_1(N-n),1)$ lies above the line $L_{\kappa^{(1)}}$. In the next section we shall describe the $(l+1)$-th step, $l \geq 1$, of the stopping time algorithm and justify the termination of the procedure.

5.3 Description of the $(l+1)$-th step, $l \geq 1$, of the stopping time procedure

Similar to the procedure described in the previous chapter we shall denote by I_{l+1} the index vector $I_{l+1} = (i_1, \ldots, i_{l+1})$. Each entry i_k, $k \geq 2$, of the vector I_{l+1} was chosen in the $(k-1)$-th step and varies in some finite range which depends on i_{k-1}. This section explains this recursion. The number i_1 is equal to one and the number i_2 corresponds to some index i from the first step of the stopping time procedure described in the previous section, and where i is the index for the zero curves $\mathcal{C}_i$ of $\partial_2\theta_{\tilde{\kappa}^{(1)}}$ in $\mathcal{H}$. In the beginning of the $(l+1)$-th step we have to deal with a finite sum of maximal operators

$$\mathcal{M}_{I_{l+1}} f(\cdot) = \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_{l+1}}(x)))\eta_{i_{l+1}}(x)dx, \quad f \geq 0.$$

The coordinates are given by

$$\Phi_{i_{l+1}}(x_1, x_2) = \left(x_1, x_2 + c_1 x_1^{\frac{n}{L}} + r^{(1)}_{i_{l+1}}(x_1^{\frac{1}{L}}), \phi_{i_{l+1}}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2\right)\right).$$

Here $r^{(1)}_{i_{l+1}}$ is a real-valued analytic function in some fractional power that satisfies

$$r^{(1)}_{i_{l+1}}(x_1) = o(x_1^n) \quad \text{for} \quad x_1 \to 0.$$

Furthermore, $\phi_{i_{l+1}}$ is also a real-valued analytic function of the form

$$\phi_{i_{l+1}}(x_1, x_2) = c_2 x_1^{NQ_{I_l}} + r^{(2)}_{i_{l+1}}(x_1) + \theta_{i_{l+1}}(x_1, x_2),$$

where $\theta_{i_{l+1}}$ is analytic and satisfies $\mathcal{T}(\theta_{i_{l+1}}) \subseteq \{(t_1, t_2) : t_2 \geq 1\}$. The analytic function $r^{(2)}_{i_{l+1}}$ satisfies $r^{(2)}_{i_{l+1}}(x_1) = o(x_1^{NQ_{I_l}})$ for $x_1 \to 0$. Similar to the previous chapter we set

$Q_{I_l} = \prod_{r=1}^{l} q_{i_r}$, where each q_{i_r} is a positive integer from the r-th step. Furthermore, the function $\theta_{i_{l+1}}$ can be decomposed as usually

$$\theta_{i_{l+1}}(x_1, x_2) = \theta_{\kappa^{(i_l)}, i_{l+1}}(x_1, x_2) + R_{\kappa^{(i_l)}, i_{l+1}}(x_1, x_2),$$

where $\theta_{\kappa^{(i_l)}, i_{l+1}}$ is the $\kappa^{(i_l)}$-homogeneous polynomial of degree one corresponding to the face of $\mathcal{N}_d(\theta_{i_{l+1}})$ lying on the line

$$L_{\kappa^{(i_l)}} = \left\{ (t_1, t_2) : \; \kappa_1^{(i_l)} t_1 + \kappa_2^{(i_l)} t_2 = 1 \right\},$$

where

$$\kappa^{(i_l)} = \left(\kappa_1^{(i_l)}, \kappa_2^{(i_l)} \right) = \left(\frac{1}{m_{i_l}}, \frac{p_{i_l}}{m_{i_l}} \right), \quad p_{i_l}, m_{i_l} \in \mathbb{N}, \quad \gcd(p_{i_l}, q_{i_l}) = 1.$$

Notice that all the notations $(p_{i_1}, q_{i_1}, m_{i_1}) = (p_1, q_1, m_1)$, $\theta_{\kappa^{(i_1)}, i_2} = \theta_{\kappa^{(1)}, i}$, $R_{\kappa^{(i_1)}, i_2} = R_{\kappa^{(1)}, i}$, $\kappa^{(1)} = \kappa^{(i_1)}$, $\theta_{i_2} = \theta_i$ coincide with the notations from the previous section. From the l-th step it is known that the right endpoint $(A_{i_{l+1}}, B_{i_{l+1}}) \in \mathbb{N}_0^2$ of $\mathcal{N}_d(\theta_{\kappa^{(i_l)}, i_{l+1}})$ satisfies $B_{i_{l+1}} \geq 2$. This also corresponds to the first step, where $B_{i_1} = B_i \geq 2$. The analytic remainder term $R_{\kappa^{(i_l)}, i_{l+1}}$ consists only of terms of higher $\kappa^{(i_l)}$-degree. The integers p_{i_j}, q_{i_j} satisfy

$$n < p < \frac{p_{i_1}}{q_{i_1}} < \frac{p_{i_2}}{q_{i_1} q_{i_2}} < \ldots < \frac{p_{i_l}}{q_{i_1} \cdot \ldots \cdot q_{i_{i_l}}} = \frac{p_{i_l}}{Q_{I_l}}. \tag{5.7}$$

We also have

$$N < \frac{m_{i_1}}{q_{i_1}} \leq m_{i_1} < \frac{m_{i_2}}{q_{i_2}} \leq m_{i_2} < \ldots < \frac{m_{i_l}}{q_{i_l}} \leq m_{i_l} = \frac{1}{\kappa_1^{(i_l)}}. \tag{5.8}$$

If the procedure did not stop in the previous step, we have in particular

$$\kappa_1^{(i_l)} Q_{I_l}(N - n) + \kappa_2^{(i_l)} > 1, \tag{5.9}$$

i.e. $(Q_{I_l}(N-n), 1)$ lies above the line $L_{\kappa^{(i_l)}}$. This also corresponds to the condition $\kappa_1^{(1)} q_1 (N - n) + \kappa_2^{(1)} > 1$ in the first step. In $\mathcal{H}$ the function

$$\eta_{i_{l+1}}(x) = \eta \left(\frac{x_2}{\varepsilon_{i_{l+1}} x_1^{\frac{p_{i_l}}{L Q_{I_l}}}} \right) \psi_{i_{l+1}}(x_1, x_2)$$

localizes the coordinates to the domain

$$\left\{(x_1,x_2)\in\mathcal{H}:\ |x_2|\le 2\varepsilon_{i_{l+1}}x_1^{\frac{p_{i_l}}{LQ_{I_l}}}\right\}\cap\operatorname{supp}\psi_{i_{l+1}},$$

and the dyadic parameter $\varepsilon_{i_{l+1}}=2^{-M_{i_{l+1}}}$, $M_{i_{l+1}}\in\mathbb{N}$, is assumed to be sufficiently small. The function $\psi_{i_{l+1}}$ is a smooth positive bump function defined on $\mathcal{H}$ and is supported in a sufficiently small neighborhood of the origin.

5.3.1 Case 1: $\mathcal{N}(\theta_{i_{l+1}})\subseteq\{(t_1,t_2):\ t_2\ge B_{i_{l+1}}\}$

This means that the Newton polyhedron $\mathcal{N}(\theta_{i_{l+1}})$ is contained in the half-plane $t_2\ge B_{i_{l+1}}$. The stopping time procedure will stop at this stage of the $(l+1)$-th step, since this situation can again be dealt with by means of the bi-dyadic decomposition.
With usual notations we can estimate $\mathcal{M}_{I_{l+1}}f$ pointwise by

$$\begin{aligned}&\sum_{j,k=K}^{\infty}\sup_{t>0}\int_{\mathcal{H}}f(\cdot-D_t^a(\Phi_{i_{l+1}}(x)))\chi_j(x_1)\chi_k(x_2)\eta_{i_{l+1}}(x)dx\\ =\ &\sum_{j,k=K}^{\infty}2^{-j-k}\sup_{t>0}\int_{\mathcal{H}}f(\cdot-D_t^a(\Phi_{i_{l+1}}(2^{-j}x_1,2^{-k}x_2)))\chi\otimes\chi(x)\eta_{i_{l+1}}(2^{-j}x_1,2^{-k}x_2)dx.\end{aligned}$$

Observe that, again, the positive integer K can be assumed to be sufficiently large if the support of $\psi_{i_{l+1}}$ is sufficiently small. With standard arguments we conclude that for $\varepsilon_{i_{l+1}}$ sufficiently small we have

$$\forall x\in\mathbb{R}^2\cap\mathcal{H}\ \forall(j,k)\in\mathbb{N}^2:\ \chi\otimes\chi(x)\eta_{i_{l+1}}(2^{-j}x_1,2^{-k}x_2)\neq 0\Longrightarrow 2^{-k+j\frac{p_{i_l}}{LQ_{I_l}}}\le\sqrt{\varepsilon_{i_{l+1}}}\ll 1,$$

and, in particular, for every $x\in\mathcal{H}\cap\operatorname{supp}\chi\otimes\chi$, we have $x_1\sim 1$ and $|x_2|\sim 1$. We get

$$\Phi_{i_{l+1}}(2^{-j}x_1,2^{-k}x_2)=\begin{pmatrix}2^{-j}x_1\\ 2^{-j\frac{n}{L}}\left(2^{-k+j\frac{n}{L}}x_2+c_1x_1^{\frac{n}{L}}+r^{(1)}_{i_{l+1},j}(x_1)\right)\\ 2^{-j\frac{N}{L}}\left(c_2x_1^{\frac{N}{L}}+r^{(2)}_{i_{l+1},j}(x_1)+2^{j\frac{N}{L}}\theta_{i_{l+1}}\left((2^{-j}x_1)^{\frac{1}{LQ_{I_l}}},2^{-k}x_2\right)\right)\end{pmatrix}^T,$$

where $r^{(1)}_{i_{l+1},j}$, $r^{(2)}_{i_{l+1},j}$ are smooth perturbations.
Next, observe that since $\mathcal{N}(\theta_{i_{l+1}}) \subseteq \{(t_1,t_2):\ t_2 \geq B_{i_{l+1}}\}$, we get

$$\begin{aligned}2^{j\frac{N}{L}}\theta_{i_{l+1}}\left((2^{-j}x_1)^{\frac{1}{LQ_{I_l}}},2^{-k}x_2\right) &= 2^{j\frac{N}{L}-j\frac{A_{i_{l+1}}}{LQ_{I_l}}-kB_{i_{l+1}}}\cdot C_{A_{i_{l+1}},B_{i_{l+1}}}\cdot x_1^{\frac{A_{i_{l+1}}}{LQ_{I_l}}}x_2^{B_{i_{l+1}}}\\ &\quad+2^{j\frac{N}{L}-j\frac{A_{i_{l+1}}}{LQ_{I_l}}-kB_{i_{l+1}}}r^{(3)}_{i_{l+1},j,k}(x_1,x_2),\end{aligned}$$

where $C_{A_{i_{l+1}},B_{i_{l+1}}} \neq 0$ and $r^{(3)}_{i_{l+1},j,k}$ is a smooth perturbation. Since $B_{i_{l+1}} \geq 2$, we conclude that for $(x_1,x_2) \in \operatorname{supp}\chi\otimes\chi\cap\mathcal{H}$

$$\left|\partial_2 x_1^{\frac{A_{i_{l+1}}}{LQ_{I_l}}}x_2^{B_{i_{l+1}}}\right| \sim 1 \sim \left|\partial_2^2 x_1^{\frac{A_{i_{l+1}}}{LQ_{I_l}}}x_2^{B_{i_{l+1}}}\right|.$$

Recall that $n \neq L$, which implies that for $x_1 \sim 1$

$$\left|\partial_1 c_1 x_1^{\frac{n}{L}}\right| \sim 1 \sim \left|\partial_1^2 c_1 x_1^{\frac{n}{L}}\right|.$$

Using (5.7) we see that

$$-k+j\frac{n}{L} \leq -k+j\frac{p}{L} \leq -k+j\frac{p_{i_l}}{LQ_{I_l}},$$

and therefore we conclude $2^{-k+j\frac{n}{L}} \leq \sqrt{\varepsilon_{i_{l+1}}} \ll 1$ for all $(j,k)\in\mathbb{N}^2$ such that

$$\chi\otimes\chi(\cdot,\cdot)\eta_{i_{l+1}}(2^{-j}\cdot,2^{-k}\cdot) \neq 0.$$

Next, we show that the parameter $2^{j\frac{N}{L}-j\frac{A_{i_{l+1}}}{LQ_{I_l}}-kB_{i_{l+1}}}$ is also small. First, notice that (5.8) implies $NQ_{I_l} < m_{i_l}$. Using $\left(A_{i_{l+1}},B_{i_{l+1}}\right) \in L_{\kappa^{(i_l)}}$ we get

$$\frac{NQ_{I_l}}{m_{i_l}} < 1 = \kappa_1^{(i_l)}A_{i_{l+1}} + \kappa_2^{(i_l)}B_{i_{l+1}} = \frac{1}{m_{i_l}}A_{i_{l+1}} + \frac{p_{i_l}}{m_{i_l}}B_{i_{l+1}}.$$

This gives

$$\frac{NQ_{I_l}-A_{i_{l+1}}}{B_{i_{l+1}}} < p_{i_l},$$

which in turn implies the estimate

$$\begin{aligned} j\frac{N}{L} - j\frac{A_{i_{l+1}}}{LQ_{I_l}} - kB_{i_{l+1}} &= B_{i_{l+1}}\left(j\frac{NQ_{I_l} - A_{i_{l+1}}}{B_{i_{l+1}}LQ_{I_l}} - k\right) \\ &< B_{i_{l+1}}\left(j\frac{p_{i_l}}{LQ_{I_l}} - k\right) \\ &\leq -B_{i_{l+1}} \cdot \frac{M_{i_{l+1}}}{2}. \end{aligned}$$

We also claim that

$$Q_{I_l}N - A_{i_{l+1}} > 0. \tag{5.10}$$

Assume that (5.10) is false. Then we get the estimate

$$Q_{I_l}(N-n) < Q_{I_l}N \leq A_{i_{l+1}}.$$

Since $B_{i_{l+1}} \geq 2$, we have $\kappa_2^{(i_l)}B_{i_{l+1}} > \kappa_2^{(i_l)}$. This gives

$$\kappa_1^{(i_l)}Q_{I_l}(N-n) + \kappa_2^{(i_l)} < \kappa_1^{(i_l)}A_{i_{l+1}} + \kappa_2^{(i_l)}B_{i_{l+1}} = 1.$$

Thus we conclude that $(Q_{I_l}(N-n), 1)$ must lie strictly below the line $L_{\kappa^{(i_l)}}$. This contradicts (5.9). Again, in view of Lemma 2.2.1 we see that it is sufficient to show that every maximal operator

$$\mathcal{M}^{j,k}_{I_{l+1}}f(\cdot) = 2^{-j-k}\sup_{t>0}\int\limits_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi^{j,k}_{i_{l+1}}(x)))\nu(x)\chi\otimes\chi(x)dx, \quad -k + j\frac{p_{i_l}}{LQ_{I_l}} \leq -\frac{M_{i_{l+1}}}{2},$$

with

$$\Phi^{j,k}_{i_{l+1}}(x_1,x_2) = \begin{pmatrix} x_1 \\ 2^{-k+j\frac{n}{L}}x_2 + c_1x_1^{\frac{n}{L}} + r^{(1)}_{i_{l+1},j}(x_1) \\ c_2x_1^{\frac{N}{L}} + r^{(2)}_{i_{l+1},j}(x_1) + 2^{j\frac{N}{L}}\theta_{i_{l+1}}\left((2^{-j}x_1)^{\frac{1}{LQ_{I_l}}}, 2^{-k}x_2\right) \end{pmatrix}^T$$

is bounded on $L^2(\mathbb{R}^3)$ with the norm at most a constant multiple of $2^{-\delta(j+k)}$ for some $\delta > 0$. The function ν is a smooth positive bump function supported in $\mathcal{H}$ and is identically one on $\chi\otimes\chi\cap\mathcal{H}$. Applying Corollary 3.3.10 we conclude that the corresponding oscillatory integral $\Lambda^{j,k}_{I_{l+1}}$ satisfies

$$\left|\nabla\Lambda^{j,k}_{I_{l+1}}(\lambda)\right| + \left|\Lambda^{j,k}_{I_{l+1}}(\lambda)\right| \lesssim 2^{-j-k}\min\left\{1, \frac{(1+|\lambda|)^{-\min\left\{\frac{1}{2}+\frac{1}{B_{i_{l+1}}},\frac{5}{6}\right\}}}{\min\left\{2^{-k+j\frac{n}{L}}, (2^{j\frac{N}{L}-j\frac{A_{i_{l+1}}}{LQ_{I_l}}-kB_{i_{l+1}}})^{\frac{1}{B_{i_{l+1}}}}\right\}}\right\}.$$

Since $NQ_{I_l} - A_{i_{l+1}} > 0$, a trivial estimate gives directly

$$\frac{1}{\min\left\{2^{-k+j\frac{n}{L}}, (2^{j\frac{N}{L}-j\frac{A_{i_{l+1}}}{LQ_{I_l}}-kB_{i_{l+1}}})^{\frac{1}{B_{i_{l+1}}}}\right\}} \leq 2^k.$$

This implies

$$\left|\nabla\Lambda^{j,k}_{I_{l+1}}(\lambda)\right| + \left|\Lambda^{j,k}_{I_{l+1}}(\lambda)\right| \lesssim 2^{-j-k}\min\left\{1, \frac{2^k}{(1+|\lambda|)^{\min\left\{\frac{1}{2}+\frac{1}{B_{i_{l+1}}},\frac{5}{6}\right\}}}\right\}.$$

Taking an appropriate geometric mean we conclude from Theorem 4.0.13 that the $L^2(\mathbb{R}^2)$-norm of every maximal operator $\mathcal{M}^{j,k}_{I_{l+1}}$ is bounded by $2^{-\delta(j+k)}$ with some $\delta > 0$. Thus the procedure stops at this stage of the $(l+1)$-th step.

5.3.2 Case 2: $\mathcal{N}(\theta_{i_{l+1}}) \not\subseteq \{(t_1,t_2):\ t_2 \geq B_{i_{l+1}}\}$

This means $\mathcal{N}_d(\theta_{i_{l+1}})$ contains further vertices lying below the line $t_2 = B_{i_{l+1}}$. Denote the closest vertex of $\mathcal{N}_d(\theta_{i_{l+1}})$ which lies below the line $t_2 = B_{i_{l+1}}$ by $(\tilde{A}_{i_{l+1}}, \tilde{B}_{i_{l+1}})$. We have $1 \leq \tilde{B}_{i_{l+1}} < B_{i_{l+1}}$. The edge $\mathfrak{E}_{i_{l+1}} = [(A_{i_{l+1}}, B_{i_{l+1}}), (\tilde{A}_{i_{l+1}}, \tilde{B}_{i_{l+1}})]$ of $\mathcal{N}_d(\theta_{i_{l+1}})$ lies on the line

$$L_{\tilde{\kappa}^{(i_{l+1})}} = \left\{(t_1,t_2):\ \tilde{\kappa}_1^{(i_{l+1})}t_1 + \tilde{\kappa}_2^{(i_{l+1})}t_2 = 1\right\},$$

with the uniquely determined weight

$$\tilde{\kappa}^{(i_{l+1})} = \left(\tilde{\kappa}_1^{(i_{l+1})}, \tilde{\kappa}_2^{(i_{l+1})}\right) = \left(\frac{q_{i_{l+1}}}{m_{i_{l+1}}}, \frac{p_{i_{l+1}}}{m_{i_{l+1}}}\right), \quad \gcd(p_{i_{l+1}}, q_{i_{l+1}}) = 1.$$

From the geometry of the Newton diagram we also conclude that the absolute value of the slope of the line $L_{\kappa^{(i_l)}}$ is larger than the absolute value of the slope of the line $L_{\tilde{\kappa}^{(i_{l+1})}}$, i.e.

$$\frac{1}{p_{i_l}} = \frac{\kappa_1^{(i_l)}}{\kappa_2^{(i_l)}} > \frac{\tilde{\kappa}_1^{(i_{l+1})}}{\tilde{\kappa}_2^{(i_{l+1})}} = \frac{q_{i_{l+1}}}{p_{i_{l+1}}}.$$

Furthermore,

$$m_{i_l} = \frac{1}{\kappa_1^{(i_l)}} < \frac{1}{\tilde{\kappa}_1^{(i_{l+1})}} = \frac{m_{i_{l+1}}}{q_{i_{l+1}}} \leq m_{i_{l+1}}.$$

Denote by $\theta_{\tilde{\kappa}^{(i_{l+1})}}$ the $\tilde{\kappa}^{(i_{l+1})}$-homogeneous part of degree one of $\theta_{i_{l+1}}$, i.e.

$$\theta_{\tilde{\kappa}^{(i_{l+1})}}(x_1, x_2) = \sum_{(\alpha,\beta)\in\mathfrak{E}_{i_{l+1}}\cap\mathbb{N}_0^2} c_{\alpha,\beta} x_1^\alpha x_2^\beta,$$

if $\theta_{i_{l+1}}(x_1, x_2) = \sum_{\alpha,\beta=0}^{\infty} c_{\alpha,\beta} x_1^\alpha x_2^\beta$. Decompose $\theta_{i_{l+1}}$ as usual

$$\theta_{i_{l+l}} = \theta_{\tilde{\kappa}^{(i_{l+1})}} + R_{\tilde{\kappa}^{(i_{l+1})}}$$

in its $\tilde{\kappa}^{(i_{l+1})}$-homogeneous part of degree one and the corresponding remainder term $R_{\tilde{\kappa}^{(i_{l+1})}}$. As in the first step we decompose the domain $|x_2| \leq 2\varepsilon_{i_{l+1}} x_1^{\frac{p_{i_l}}{LQ_{I_l}}}$ in $\mathcal{H}$ into two domains

$$T_{i_{l+1}} = \left\{ (x_1, x_2) \in \mathcal{H} : \; N_{i_{l+1}} x_1^{\frac{p_{i_{l+1}}}{LQ_{I_l} q_{i_{l+1}}}} < |x_2| \leq 2\varepsilon_{i_{l+1}} x_1^{\frac{p_{i_l}}{LQ_{I_l}}} \right\},$$

$$H_{i_{l+1}} = \left\{ (x_1, x_2) \in \mathcal{H} : \; |x_2| \leq N_{i_{l+1}} x_1^{\frac{p_{i_{l+1}}}{LQ_{I_l} q_{i_{l+1}}}} \right\},$$

where the number $N_{i_{l+1}} \in \mathbb{N}$ is assumed to be very large. In the transition domain $T_{i_{l+1}}$ the procedure terminates, since the L^2-estimate is again obtained by means of the bi-dyadic decomposition as in the previous case. Localize the coordinates to the homogeneous domain $H_{i_{l+1}}$ setting

$$\eta_{\tilde{\kappa}^{(i_{l+1})}}(x_1, x_2) = \eta \left(\frac{x_2}{N_{i_{l+1}} x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}} \right) \psi_{i_{l+1}}(x_1, x_2),$$

and

$$Q_{I_{l+1}} = \prod_{r=1}^{l+1} q_{i_r}.$$

It is enough to show that the maximal operator

$$\mathcal{M}_{\tilde{\kappa}^{(i_{l+1})}} f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_{l+1}}(x)))\eta_{\tilde{\kappa}^{(i_{l+1})}}(x)dx, \quad f \geq 0,$$

is bounded on $L^2(\mathbb{R}^3)$. Again, different cases can occur and these cases depend on the position of the point $(Q_{I_l}(N-n), 1)$ with respect to the line $L_{\tilde{\kappa}^{(i_{l+1})}}$. The procedure again does not stop in this step if and only if the point $(Q_{I_l}(N-n), 1)$ lies above the line $L_{\tilde{\kappa}^{(i_{l+1})}}$ and the polynomial $\partial_2 \theta_{\tilde{\kappa}^{(i_{l+1})}}$ has zeros in $\mathcal{H}$.

Case 2.1: $\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}(N-n) + \tilde{\kappa}_2^{(i_{l+1})} \leq 1$

As in the first step we decompose dyadically with respect to the weight $(LQ_{I_l}\tilde{\kappa}_1^{(i_{l+1})}, \tilde{\kappa}_2^{(i_{l+1})})$, cf. Lemma A.0.2. Set

$$l_j = -j(LQ_{I_l}\tilde{\kappa}_1^{(i_{l+1})} + \tilde{\kappa}_2^{(i_{l+1})}).$$

We obtain the usual pointwise estimate

$$\mathcal{M}_{\tilde{\kappa}^{(i_{l+1})}} f(\cdot) \leq \sum_{j=K}^{\infty} 2^{l_j} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D_t^a(\Phi_{i_{l+1}}(\delta_{2^{-j}}(x))))\mu(x)\rho(x)\eta_{\tilde{\kappa}^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx,$$

where the positive integer K is large, since the support of $\psi_{i_{l+1}}$ is small. In $\mathcal{H}$ the amplitude

$$x \longmapsto \rho(x)\eta_{\tilde{\kappa}^{(i_{l+1})}}(\delta_{2^{-j}}(x)) = \rho(x)\eta\left(\frac{x_2}{N_{i_{l+1}} x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}\right)\psi_{i_{l+1}}(\delta_{2^{-j}}(x))$$

is supported in

$$\{(x_1, x_2) \in \mathcal{H} : x_1 \sim 1, \, |x_2| \lesssim 1\},$$

and μ is a suitable positive bump function with compact support in $\mathcal{H}$ and identically one on the above set. For every j we rewrite the functions $\Phi_{i_{l+1}}(\delta_{2^{-j}}(\cdot))$ as

$$\Phi_{i_{l+1}}(\delta_{2^{-j}}(x)) = \begin{pmatrix} 2^{-j\tilde{\kappa}_1^{(i_{l+1})}LQ_{I_l}} & 0 & 0 \\ 0 & 2^{-j\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n} & 0 \\ 0 & 0 & 2^{-j\tilde{\kappa}_1^{(i_{l+1})}NQ_{I_l}} \end{pmatrix} \cdot \Phi^j_{i_{l+1}}(x),$$

where

$$\Phi^j_{i_{l+1}}(x) = \begin{pmatrix} x_1 \\ 2^{-j(\tilde{\kappa}_2^{(i_{l+1})}-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}x_2 + c_1 x_1^{\frac{n}{L}} + R_1^j(x_1) \\ c_2 x_1^{\frac{N}{L}} + R_2^j(x_1) + 2^{-j(1-\tilde{\kappa}_1^{(i_{l+1})}NQ_{I_l})}\left(\theta_{\tilde{\kappa}^{(i_{l+1})}}(x_1^{\frac{1}{LQ_{I_l}}}, x_2) + R_3^j(x)\right) \end{pmatrix},$$

and the functions R_1^j, R_2^j and R_3^j are smooth perturbations.
After applying Lemma 2.2.1 we reduce the L^2-estimate of the maximal operator $\mathcal{M}_{\tilde{\kappa}^{(i_{l+1})}}$ to the sum of maximal operators

$$\mathcal{M}^j_{\tilde{\kappa}^{(i_{l+1})}}f(\cdot) = 2^{l_j}\sup_{t>0}\int_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi^j_{i_{l+1}}(x)))\mu(x)\rho(x)\eta_{\tilde{\kappa}^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx, \quad j \geq K.$$

We shall prove that every maximal operator $\mathcal{M}^j_{\tilde{\kappa}^{(i_{l+1})}}$ is bounded on $L^2(\mathbb{R}^3)$ with the norm at most a constant multiple of $2^{-j\delta}$ for some $\delta > 0$. Because of (5.7), we obtain

$$n < p < \frac{p_{i_l}}{Q_{I_l}} < \frac{p_{i_{l+1}}}{Q_{I_{l+1}}} = \frac{\tilde{\kappa}_2^{(i_{l+1})}}{\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}}.$$

This yields $\tilde{\kappa}_2^{(i_{l+1})} - \tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n > 0$, i.e. $2^{-j(\tilde{\kappa}_2^{(i_{l+1})}-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}$ is a small parameter.
As already observed in the previous case, (5.8) implies $NQ_{I_l} < m_{i_l}$. From the geometry of the Newton polyhedron $\mathcal{N}(\theta_{i_{l+1}})$ we conclude that $m_{i_l} < \frac{1}{\tilde{\kappa}^{(i_{l+1})}}$. This implies that the parameter $2^{-j(1-\tilde{\kappa}_1^{(i_{l+1})}NQ_{I_l})}$ is also very small. From the identity

$$\frac{2^{-j(1-\tilde{\kappa}_1^{(i_{l+1})}NQ_{I_l})}}{2^{-j(\tilde{\kappa}_2^{(i_{l+1})}-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}} = 2^{j(\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}(N-n)+\tilde{\kappa}_2^{(i_{l+1})}-1)} \tag{5.11}$$

we see that the quotient on the left hand-side is either equal to one, if the point $(Q_{I_l}(N-n),1)$ lies on the line $L_{\tilde{\kappa}^{(i_{l+1})}}$, or very small if $\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}(N-n)+\tilde{\kappa}_2^{(i_{l+1})}<1$. Also observe that since $n\neq L$, we always have

$$\left|\partial_1 x_1^{\frac{n}{L}}\right| \sim 1 \sim \left|\partial_1^2 x_1^{\frac{n}{L}}\right|$$

if $x_1\sim 1$. Furthermore,

$$\left|\partial_2^{B_{i_{l+1}}}\theta_{\tilde{\kappa}^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}},x_2\right)\right|\sim 1$$

also holds true for $x_1\sim 1$. First, we shall discuss the case where $(Q_{I_l}(N-n),1)$ lies below the line $L_{\tilde{\kappa}^{(i_{l+1})}}$.

Case 2.1.1: $\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}(N-n)+\tilde{\kappa}_2^{(i_{l+1})}<1$

In this case the quotient (5.11) is very small, since K is very large. If we apply Lemma 3.3.7, we conclude that the oscillatory integral $\Lambda^j_{\tilde{\kappa}^{(i_{l+1})}}$ corresponding to the maximal operator $\mathcal{M}^j_{\tilde{\kappa}^{(i_{l+1})}}$ can be estimated by

$$\left|\nabla\Lambda^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right|+\left|\Lambda^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right|\lesssim\frac{2^{l_j}\cdot(1+|\lambda|)^{-\frac{1}{2}-\frac{1}{B_{i_{l+1}}}}}{\min\left\{2^{-j\frac{1-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}N}{B_{i_{l+1}}}},2^{-j(\tilde{\kappa}_2^{(i_{l+1})}-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}\right\}}$$

if the integer K is large enough. We conclude

$$\left|\nabla\Lambda^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right|+\left|\Lambda^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right|\lesssim\frac{2^{l_j}\cdot\max\left\{2^{j\frac{1-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}N}{B_{i_{l+1}}}},2^{j(\tilde{\kappa}_2^{(i_{l+1})}-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}\right\}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B_{i_{l+1}}}}}.$$

Observe that because of (5.10) we get

$$1=\tilde{\kappa}_1^{(i_{l+1})}A_{i_{l+1}}+\tilde{\kappa}_2^{(i_{l+1})}B_{i_{l+1}}\leq\tilde{\kappa}_1^{(i_{l+1})}NQ_{I_l}+\tilde{\kappa}_2^{(i_{l+1})}B_{i_{l+1}}.$$

This implies

$$\frac{1-\tilde{\kappa}_1^{(i_{l+1})}NQ_{I_l}}{B_{i_{l+1}}}\leq\tilde{\kappa}_2^{(i_{l+1})}.$$

Combining all these observations we get

$$\begin{aligned}\left|\nabla\Lambda^j_{\tilde\kappa^{(i_{l+1})}}(\lambda)\right|+\left|\Lambda^j_{\tilde\kappa^{(i_{l+1})}}(\lambda)\right| &\lesssim 2^{-j(LQ_{I_l}\tilde\kappa_1^{(i_{l+1})}+\tilde\kappa_2^{(i_{l+1})})}\cdot\frac{2^{j\tilde\kappa_2^{(i_{l+1})}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B_{i_{l+1}}}}}\\ &\lesssim \frac{2^{-jLQ_{I_l}\tilde\kappa_1^{(i_{l+1})}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B_{i_{l+1}}}}}.\end{aligned}$$

If we apply Theorem 4.0.13, we conclude that $\|\mathcal{M}^j_{\tilde\kappa^{(i_{l+1})}}\|_{L^2\to L^2}\lesssim 2^{-jLQ_{I_l}\tilde\kappa_1^{(i_{l+1})}}$. The desired L^2-boundedness of $\mathcal{M}_{\tilde\kappa^{(i_{l+1})}}$ follows from Minkowski's inequality.

Case 2.1.2: $\tilde\kappa_1^{(i_{l+1})}Q_{I_l}(N-n)+\tilde\kappa_2^{(i_{l+1})}=1$

This means that the ratio (5.11) is identically one. The zeros of the polynomial $\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}$ in $\mathcal{H}$ (if at all existent) are finitely many curves

$$\mathcal{C}_\alpha=\left\{\left(x_1,z_\alpha x_1^{\frac{p_{i_{l+1}}}{q_{i_{l+1}}}}\right)\,:\;x_1>0\right\},\quad z_\alpha\in\mathbb{R}.$$

The index α varies in some finite range which depends on i_{l+1}. For small $\varepsilon_\alpha>0$ we localize near the zeros of $\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}\left(\cdot^{\frac{1}{LQ_{I_l}}},\cdot\right)$ by the function

$$\eta^\alpha_{\tilde\kappa^{(i_{l+1})}}(x_1,x_2)=\eta\left(\frac{x_2-z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}{\varepsilon_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}\right),$$

with η defined in (4.8). We get

$$\mathcal{M}^j_{\tilde\kappa^{(i_{l+1})}}f\le\sum_\alpha\mathcal{M}^j_{\tilde\kappa^{(i_{l+1})},\alpha}f+\overline{\mathcal{M}}^j_{\tilde\kappa^{(i_{l+1})}}f,$$

where

$$\mathcal{M}^j_{\tilde\kappa^{(i_{l+1})},\alpha}f(\cdot)=2^{l_j}\sup_{t>0}\int_{\mathbb{R}^2}f(\cdot-D^a_t(\Phi^j_{i_{l+1}}(x)))\mu(x)\rho(x)\eta^\alpha_{\tilde\kappa^{(i_{l+1})}}(x)\eta_{\tilde\kappa^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx,$$

$$\overline{\mathcal{M}}^j_{\tilde\kappa^{(i_{l+1})}}f(\cdot) = 2^{l_j}\sup_{t>0}\int\limits_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi^j_{i_{l+1}}(x)))\mu(x)\rho(x)(1-\sum_\alpha \eta^\alpha_{\tilde\kappa^{(i_{l+1})}}(x))\eta_{\tilde\kappa^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx.$$

Since $\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}\right) \equiv 0$ for $x_1 > 0$ and

$$\left|\partial_2^{B_{i_{l+1}}}\theta_{\tilde\kappa^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2\right)\right| \sim 1$$

for $(x_1, x_2) \in \operatorname{supp}\mu$, we conclude from Lemma 3.3.7 that the oscillatory integral $\Lambda^j_{\tilde\kappa^{(i_{l+1})},\alpha}$ corresponding to the maximal operator $\mathcal{M}^j_{\tilde\kappa^{(i_{l+1})},\alpha}$ can be estimated by

$$\left|\nabla\Lambda^j_{\tilde\kappa^{(i_{l+1})},\alpha}(\lambda)\right| + \left|\Lambda^j_{\tilde\kappa^{(i_{l+1})},\alpha}(\lambda)\right| \lesssim \frac{2^{l_j}\cdot(1+|\lambda|)^{-\frac{1}{2}-\frac{1}{B_{i_{l+1}}}}}{\min\left\{2^{-j\frac{1-\tilde\kappa_1^{(i_{l+1})}NQ_{I_l}}{B_{i_{l+1}}}}, 2^{-j(\tilde\kappa_2^{(i_{l+1})}-\tilde\kappa_1^{(i_{l+1})}Q_{I_l}n)}\right\}},$$

provided ε_α is small enough and K is large. Eventually, we get

$$\left|\nabla\Lambda^j_{\tilde\kappa^{(i_{l+1})},\alpha}(\lambda)\right| + \left|\Lambda^j_{\tilde\kappa^{(i_{l+1})},\alpha}(\lambda)\right| \lesssim \frac{2^{-jLQ_{I_l}\tilde\kappa_1^{(i_{l+1})}}}{(1+|\lambda|)^{\frac{1}{2}+\frac{1}{B_{i_{l+1}}}}},$$

which implies that $\|\mathcal{M}^j_{\tilde\kappa^{(i_{l+1})},\alpha}\|_{L^2\to L^2} \lesssim 2^{-jLQ_{I_l}\tilde\kappa_1^{(i_{l+1})}}$.

In order to estimate $\overline{\mathcal{M}}^j_{\tilde\kappa^{(i_{l+1})}}$, we shall again distinguish two different cases. Since $\tilde\kappa_2^{(i_{l+1})} \le \frac{1}{2}$, the zeros of $\partial_2^2\theta_{\tilde\kappa^{(i_{l+1})}}$ in $\mathcal{H}$ (if at all existent) are finitely many curves

$$\overline{\mathcal{C}}_\beta = \left\{\left(x_1, \overline{z}_\beta x_1^{\frac{p_{i_{l+1}}}{q_{i_{l+1}}}}\right) : x_1 > 0\right\}, \quad \overline{z}_\beta \in \mathbb{R}.$$

The function

$$\overline{\eta}^\beta_{\tilde\kappa^{(i_{l+1})}}(x_1,x_2) = \eta\left(\frac{x_2 - \overline{z}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}{\overline{\varepsilon}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}\right)\mu(x)\rho(x)(1-\sum_\alpha \eta^\alpha_{\tilde\kappa^{(i_{l+1})}}(x))$$

localizes the coordinates near the zeros of $\partial_2^2\theta_{\tilde{\kappa}^{(i_{l+1})}}\left(\cdot^{\frac{1}{LQ_{I_l}}},\cdot\right)$. Every parameter $\bar{\varepsilon}_\beta > 0$ is small. As before, set

$$\overline{\overline{\eta}}_{\tilde{\kappa}^{(i_{l+1})}}(x_1,x_2) = \left(1 - \sum_\beta \eta\left(\frac{x_2 - \bar{z}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}{\bar{\varepsilon}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}\right)\right)\mu(x)\rho(x)(1 - \sum_\alpha \eta^\alpha_{\tilde{\kappa}^{(i_{l+1})}}(x)),$$

and estimate

$$\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})}} f \le \sum_\beta \overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})},\beta} f + \overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(i_{l+1})}} f,$$

with

$$\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})},\beta} f(\cdot) = 2^{l_j} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi^j_{i_{l+1}}(x)))\overline{\eta}^\beta_{\tilde{\kappa}^{(i_{l+1})}}(x)\eta_{\tilde{\kappa}^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx,$$

$$\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(i_{l+1})}} f(\cdot) = 2^{l_j} \sup_{t>0} \int_{\mathbb{R}^2} f(\cdot - D^a_t(\Phi^j_{i_{l+1}}(x)))\overline{\overline{\eta}}_{\tilde{\kappa}^{(i_{l+1})}}(x)\eta_{\tilde{\kappa}^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx.$$

Since for every $x \in \operatorname{supp} \overline{\overline{\eta}}_{\tilde{\kappa}^{(i_{l+1})}}$ we have

$$\partial_2\theta_{\tilde{\kappa}^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2\right) \neq 0, \quad \partial_2^2\theta_{\tilde{\kappa}^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2\right) \neq 0,$$

from Corollary 3.3.10 we get an estimate for the oscillatory integral $\overline{\overline{\Lambda}}^j_{\tilde{\kappa}^{(i_{l+1})}}$ corresponding to the maximal operator $\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(i_{l+1})}}$

$$\begin{aligned} \left|\nabla\overline{\overline{\Lambda}}^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right| + \left|\overline{\overline{\Lambda}}^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right| &\lesssim \frac{2^{l_j}\cdot(1+|\lambda|)^{-\min\{\frac{1}{2}+\frac{1}{B_{i_{l+1}}},\frac{5}{6}\}}}{\min\left\{2^{-j\frac{1-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}N}{B_{i_{l+1}}}}, 2^{-j(\tilde{\kappa}_2^{(i_{l+1})}-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}\right\}} \\ &\lesssim \frac{2^{-jLQ_{I_l}\tilde{\kappa}^{(i_{l+1})}}}{(1+|\lambda|)^{\min\{\frac{1}{2}+\frac{1}{B_{i_{l+1}}},\frac{5}{6}\}}} \end{aligned}$$

if the integer K is sufficiently large. This implies that $\|\overline{\overline{\mathcal{M}}}^j_{\tilde{\kappa}^{(i_{l+1})}}\| \lesssim 2^{-jLQ_{I_l}\tilde{\kappa}^{(i_{l+1})}}$.
In order to estimate every maximal operator $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})},\beta}$, we first observe that if for some β

we have $\overline{z}_\beta \neq z_\alpha$ for every α, then by Lemma 2.2.4 we obtain

$$\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, \overline{z}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}\right) \neq 0, \quad \partial_1\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, \overline{z}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}\right) \neq 0,$$

$$\partial_2^2\theta_{\tilde\kappa^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, \overline{z}_\beta x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}\right) = 0$$

for every $x_1 > 0$. Recall that $\partial_2^{B_{i_{l+1}}}\theta_{\tilde\kappa^{(i_{l+1})}}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2\right) \neq 0$ for $x_1 \sim 1$.

Therefore applying Lemma 3.3.12, we see that the oscillatory integral $\overline{\Lambda}^j_{\tilde\kappa^{(i_{l+1})},\beta}$ can be estimated by

$$\left|\nabla\overline{\Lambda}^j_{\tilde\kappa^{(i_{l+1})},\beta}(\lambda)\right| + \left|\overline{\Lambda}^j_{\tilde\kappa^{(i_{l+1})},\beta}(\lambda)\right| \lesssim \frac{2^{l_j}}{(1+|\lambda|)^{\frac{1}{2}+\delta}2^{-j(\tilde\kappa_2^{(i_{l+1})}-\tilde\kappa_1^{(i_{l+1})}Q_{I_l}n)}},$$

if the numbers δ, $\overline{\varepsilon}_\beta$ are chosen sufficiently small and the integer K is sufficiently large. This implies

$$\left|\nabla\overline{\Lambda}^j_{\tilde\kappa^{(i_{l+1})},\beta}(\lambda)\right| + \left|\overline{\Lambda}^j_{\tilde\kappa^{(i_{l+1})},\beta}(\lambda)\right| \lesssim \frac{2^{-jLQ_{I_l}\tilde\kappa_1^{(i_{l+1})}}}{(1+|\lambda|)^{\frac{1}{2}+\delta}}.$$

Combining all these estimates we conclude that if $\tilde\kappa_1^{(i_{l+1})}Q_{I_l}(N-n) + \tilde\kappa_2^{(i_{l+1})} \leq 1$, then $\mathcal{M}_{\tilde\kappa^{(i_{l+1})}}$ is bounded on $L^2(\mathbb{R}^3)$.

Case 2.2: $\tilde\kappa_1^{(i_{l+1})}Q_{I_l}(N-n) + \tilde\kappa_2^{(i_{l+1})} > 1$

As in the first step we shall proceed to the next step near the zeros of $\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}(\cdot^{\frac{1}{LQ_{I_l}}}, \cdot)$. More precisely, as described in the first step of the stopping time procedure, we first localize near the zeros of $\partial_2\theta_{\tilde\kappa^{(i_{l+1})}}(\cdot^{\frac{1}{LQ_{I_l}}}, \cdot)$ by means of the functions $\eta^\alpha_{\tilde\kappa^{(i_{l+1})}}$. In the remainder domain we decompose dyadically with respect to the weight $(LQ_{I_l}\tilde\kappa_1^{(i_{l+1})}, \tilde\kappa_2^{(i_{l+1})})$, and then apply the usual rescaling argument in order to reduce the L^2-estimate of $\mathcal{M}_{\tilde\kappa^{(i_{l+1})}}$ to the sum of maximal operators

$$\sum_\alpha \mathcal{M}_{\tilde\kappa^{(i_{l+1})},\alpha} + \sum_{j=K}^\infty \overline{\mathcal{M}}^j_{\tilde\kappa^{(i_{l+1})}},$$

where

$$\mathcal{M}_{\tilde{\kappa}^{(i_{l+1})},\alpha} f(\cdot) = \sup_{t>0} \int\limits_{\mathcal{H}} f(\cdot - D_t^a(\Phi_{i_{l+1}}(x)))\eta^{\alpha}_{\tilde{\kappa}^{(i_{l+1})}}(x)\psi_{i_{l+1}}(x)dx,$$

and $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})}}$ are the maximal operators from the previous case given by

$$\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})}} f(\cdot) = 2^{l_j} \sup_{t>0} \int\limits_{\mathbb{R}^2} f(\cdot - D_t^a(\Phi^j_{i_{l+1}}(x)))\mu(x)\rho(x)(1 - \sum_{\alpha} \eta^{\alpha}_{\tilde{\kappa}^{(i_{l+1})}}(x))\eta_{\tilde{\kappa}^{(i_{l+1})}}(\delta_{2^{-j}}(x))dx.$$

On the support of $\mu\rho(1 - \sum_{\alpha} \eta^{\alpha}_{\tilde{\kappa}^{(i_{l+1})}})$ the function $\partial_2 \theta_{\tilde{\kappa}^{(i_{l+1})}}(\cdot^{\frac{1}{LQ_{I_l}}}, \cdot)$ does not vanish. Furthermore, by the assumption of this case the ratio

$$\frac{2^{-j(1-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}N)}}{2^{-j(\tilde{\kappa}_2^{(i_{l+1})} - nQ_{I_l}\tilde{\kappa}_1^{(i_{l+1})})}}$$

is arbitrarily large, since we are free to choose K large enough.
Using Lemma 3.3.11 we obtain an estimate for the decay of the oscillatory integral $\overline{\Lambda}^j_{\tilde{\kappa}^{(i_{l+1})}}$ corresponding to the maximal operator $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})}}$

$$\left|\nabla\overline{\Lambda}^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right| + \left|\overline{\Lambda}^j_{\tilde{\kappa}^{(i_{l+1})}}(\lambda)\right| \lesssim \frac{2^{l_j} \cdot (1+|\lambda|)^{-\frac{1}{2} - \frac{1}{B_{i_{l+1}}}}}{\min\left\{2^{-j\frac{1-\tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}N}{B_{i_{l+1}}}}, 2^{-j(\tilde{\kappa}_2^{(i_{l+1})} - \tilde{\kappa}_1^{(i_{l+1})}Q_{I_l}n)}\right\}},$$

and usual arguments imply the L^2-boundedness of $\overline{\mathcal{M}}^j_{\tilde{\kappa}^{(i_{l+1})}}$ with the norm at most a constant multiple of $2^{-jLQ_{I_l}\tilde{\kappa}_1^{(i_{l+1})}}$.
In order to deal with every maximal operator $\mathcal{M}_{\tilde{\kappa}^{(i_{l+1})},\alpha}$, we proceed to the next step changing variables

$$\zeta_\alpha(x_1, x_2) = \left(x_1, x_2 - z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}\right), \quad (x_1, x_2) \in \mathcal{H}.$$

Recall that α is the index for the zeros of $\partial_2\theta_{\tilde{\kappa}^{(i_{l+1})}}$ in $\mathcal{H}$ and varies in some finite range depending on i_{l+1}. Clearly, the described procedure terminates if $\partial_2\theta_{\tilde{\kappa}^{(i_{l+1})}}$ does not vanish in $\mathcal{H}$. We get

$$\Phi_\alpha(x_1,x_2) = \Phi_{i_{l+1}}(\zeta_\alpha^{-1}(x_1,x_2)) = \begin{pmatrix} x_1 \\ x_2 + c_1 x_1^{\frac{n}{L}} + r^{(1)}_{i_{l+1}}(x_1^{\frac{1}{L}}) + z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}} \\ \phi_\alpha\left(x_1^{\frac{1}{LQ_{I_{l+1}}}}, x_2\right) \end{pmatrix}^T,$$

where

$$\phi_\alpha\left(x_1^{\frac{1}{LQ_{I_{l+1}}}}, x_2\right) = \phi_{i_{l+1}}\left(\zeta_\alpha^{-1}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2\right)\right) = \phi_{i_{l+1}}(x_1^{\frac{1}{LQ_{I_l}}}, x_2 + z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}).$$

Set

$$r_\alpha^{(1)}(x_1) = r^{(1)}_{i_{l+1}}(x_1) + z_\alpha x_1^{\frac{p_{i_{l+1}}}{Q_{I_{l+1}}}}.$$

The function $r_\alpha^{(1)}$ is a real-valued analytic function in some fractional power, since by assumption $r^{(1)}_{i_{l+1}}$ is a real-valued analytic function in some fractional power and $z_\alpha \in \mathbb{R}$. Notice that in view of (5.7) we get

$$n < \frac{p_{i_l}}{Q_{I_l}} < \frac{p_{i_{l+1}}}{q_{i_{l+1}} Q_{I_l}} = \frac{p_{i_{l+1}}}{Q_{I_{l+1}}},$$

and therefore $r_\alpha^{(1)}(x_1) = o(x_1^n)$. Furthermore, after the change of coordinates the density function

$$\mathcal{H} \ni (x_1,x_2) \longmapsto \eta_\alpha(x_1,x_2) = \eta\left(\frac{x_2}{\varepsilon_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}}\right) \underbrace{\psi_{i_{l+1}}(x_1, x_2 + z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}})}_{\psi_\alpha(x_1,x_2)}$$

localizes the coordinates to the small homogeneous domain $|x_2| \le 2\varepsilon_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}$ near the origin in $\mathcal{H}$. The number $\varepsilon_\alpha = 2^{-M_\alpha}$, $M_\alpha \in \mathbb{N}$, is some small dyadic parameter. We have

$$\phi_\alpha(x_1^{\frac{1}{LQ_{I_{l+1}}}}, x_2) = c_2 x_1^{\frac{N}{L}} + o(x_1^{\frac{N}{L}}) + \theta_{i_{l+1}}\left(x_1^{\frac{1}{LQ_{I_l}}}, x_2 + z_\alpha x_1^{\frac{p_{i_{l+1}}}{LQ_{I_{l+1}}}}\right).$$

Let $\tilde{\theta}_\alpha$ be the analytic function given by

$$\tilde{\theta}_\alpha(x_1, x_2) = \theta_{i_{l+1}}(x_1^{q_{i_{l+1}}}, x_2 + z_\alpha x_1^{p_{i_{l+1}}}).$$

We decompose $\tilde{\theta}_\alpha$ in

$$\tilde{\theta}_\alpha(x_1, x_2) = \tilde{\theta}_\alpha(x_1, 0) + \theta_\alpha(x_1, x_2).$$

Then $\mathcal{T}(\theta_\alpha) \subseteq \{(t_1, t_2) : \ t_2 \geq 1\}$. Lemma 2.3.1 implies that $\mathcal{N}_d(\theta_\alpha)$ contains a face $[(q_{i_{l+1}} A_{i_{l+1}}, B_{i_{l+1}}), (A_\alpha, B_\alpha)]$, $B_{i_{l+1}} \geq B_\alpha \geq 2$, lying on the line

$$L_{\kappa^{(i_{l+1})}} = \left\{(t_1, t_2) : \ \kappa_1^{(i_{l+1})} t_1 + \kappa_2^{(i_{l+1})} t_2 = 1\right\},$$

with the usual notation

$$\kappa^{(i_{l+1})} = \left(\kappa_1^{(i_{l+1})}, \kappa_2^{(i_{l+1})}\right) = \left(\frac{\tilde{\kappa}_1^{(i_{l+1})}}{q_{i_{l+1}}}, \tilde{\kappa}_2^{(i_{l+1})}\right) = \left(\frac{1}{m_{i_{l+1}}}, \frac{p_{i_{l+1}}}{m_{i_{l+1}}}\right).$$

It is possible that the face $[(q_{i_{l+1}} A_{i_{l+1}}, B_{i_{l+1}}), (A_\alpha, B_\alpha)]$ even degenerates to a vertex. The function θ_α can be decomposed in the usual way

$$\theta_\alpha = \theta_{\kappa^{(i_{l+1})},\alpha} + R_{\kappa^{(i_{l+1})},\alpha},$$

where $\theta_{\kappa^{(i_{l+1})},\alpha}$ is the $\kappa^{(i_{l+1})}$-homogeneous part of θ_α of degree one. Notice that the point $\big(Q_{I_{l+1}}(N-n), 1\big)$ lies strictly above the line $L_{\kappa^{(i_{l+1})}}$. Using (5.8) we get

$$m_{i_{l+1}} \geq \frac{m_{i_{l+1}}}{q_{i_{l+1}}} > \frac{1}{\kappa_1^{(i_{l+1})}} > N.$$

Since $\tilde{\theta}_\alpha(x_1, 0) = \mathcal{O}(x_1^{m_{i_{l+1}}})$, we have in particular, $\tilde{\theta}_\alpha(x_1, 0) = \mathcal{O}(x_1^N)$. Eventually, we conclude that

$$\phi_\alpha\left(x_1^{\frac{1}{LQ_{I_{l+1}}}}, x_2\right) = c_2 x_1^{\frac{N}{L}} + r_\alpha^{(2)}(x_1^{\frac{1}{L}}) + \theta_{\kappa^{(i_{l+1})},\alpha}\left(x_1^{\frac{1}{LQ_{I_{l+1}}}}, x_2\right) + R_{\kappa^{(i_{l+1})},\alpha}\left(x_1^{\frac{1}{LQ_{I_{l+1}}}}, x_2\right),$$

with $r_\alpha^{(2)}(x_1) = o(x_1^N)$. Thus the description of the $(l+1)$-th step is now complete and we see that we only have to proceed to the next step if $\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}(N-n) + \tilde{\kappa}_2^{(i_{l+1})} > 1$ and $\partial_2 \theta_{\tilde{\kappa}^{(i_{l+1})}}$ vanishes in $\mathcal{H}$. Thus in the $(l+2)$-th step we will have to estimate finitely many maximal operators

$$\mathcal{M}_{I_l,\alpha}f(\cdot) = \sup_{t>0} \int_{\mathcal{H}} f(\cdot - D_t^a(\Phi_\alpha(x)))\eta_\alpha(x)dx.$$

The proof of Theorem 1.3.1 is complete, if we can prove that the stopping time procedure terminates after finitely many steps. In order to prove it, assume that there is a sequence $(\tilde{\kappa}^{(i_j)})_j$ such that

$$\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}(N-n) + \tilde{\kappa}_2^{(i_{l+1})} > 1 \tag{5.12}$$

holds true for every $l \in \mathbb{N}$, i.e. in every step the point $(Q_{I_l}(N-n), 1)$ lies above the supporting line $L_{\tilde{\kappa}^{(i_{l+1})}}$ of $\mathcal{N}_d(\theta_{i_{l+1}})$. Recall that the point $(A_{i_{l+1}}, B_{i_{l+1}})$ lies on the line $L_{\tilde{\kappa}^{(i_{l+1})}}$ and $B_{i_{l+1}} \geq 2$. Since $(B_{i_l})_l$ is a decreasing sequence, it must eventually become constant, which in turn implies that the numbers q_{i_l} must eventually become one (cf. Lemma 2.3.1). We get

$$1 = \tilde{\kappa}_1^{(i_{l+1})} A_{i_{l+1}} + \tilde{\kappa}_2^{(i_{l+1})} B_{i_{l+1}} \geq 2\tilde{\kappa}_2^{(i_{l+1})},$$

which implies that $\tilde{\kappa}_2^{(i_{l+1})} \leq \frac{1}{2}$. Inequality (5.12) implies

$$\frac{\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}}{\tilde{\kappa}_2^{(i_{l+1})}}(N-n) + 1 > \frac{1}{\tilde{\kappa}_2^{(i_{l+1})}} \geq 2.$$

This gives

$$\frac{\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}}{\tilde{\kappa}_2^{(i_{l+1})}}(N-n) > 1.$$

The contradiction follows from the observation that

$$\frac{\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}}{\tilde{\kappa}_2^{(i_{l+1})}} \longrightarrow 0, \quad \text{when} \quad l \longrightarrow \infty,$$

since

$$\frac{\tilde{\kappa}_1^{(i_{l+1})} Q_{I_l}}{\tilde{\kappa}_2^{(i_{l+1})}} = \frac{\prod_{j=1}^{l+1} q_{i_r}}{p_{i_{l+1}}},$$

and $p_{i_l} \longrightarrow \infty$. The proof of Theorem 1.3.1 is now complete.

6 Averages over smooth non-analytic hypersurfaces

6.1 Smooth flat case

We construct a smooth curve in $\mathbb{R}^2$ which is not analytic at the origin and the corresponding maximal operator is only bounded on $L^\infty(\mathbb{R}^2)$.

Example 6.1.1. Let $\chi \in C_0^\infty(\mathbb{R})$, $0 \le \chi \le 1$, $\operatorname{supp}\chi \subseteq [-2,2]$ and $\chi = 1$ on $[-1,1]$. For $k \in \mathbb{N}$ let

$$I_k = [2^{-k-1}, 2^{-k})$$

be the dyadic interval of the length 2^{-k-1}.
Denote by $c(I_k)$ the center of I_k, i.e. $c(I_k) = \frac{3}{4} \cdot 2^{-k}$. Set

$$\chi_k(x) = 2^{-2^k} \chi\left(\frac{x - c(I_k)}{2^{-k-10}}\right).$$

The function χ_k is supported in the interval $[c(I_k) - 2^{-k-9}, c(I_k) + 2^{-k-9}]$ and is identically 2^{-2^k} on the smaller interval

$$\tilde{I}_k = [c(I_k) - 2^{-k-10}, c(I_k) + 2^{-k-10}]. \tag{6.1}$$

Set

$$\gamma(x) = \sum_{k=1}^{\infty} \chi_k(x), \quad x \in \mathbb{R}. \tag{6.2}$$

It is easy to see that γ is smooth and that every derivative of γ vanishes at the origin.

We claim that for every neighborhood U of the origin the maximal operator

$$\mathcal{M}f(z) = \sup_{t>0} \left| \int\limits_U f(z - t(u, \gamma(u)))du \right|, \quad z \in \mathbb{R}^2,$$

is unbounded on $L^p(\mathbb{R}^2)$ for every finite p. In fact, this is easily seen since the tangent line to the curve γ at the point $(c(I_k), \gamma(c(I_k)))$ is equal to $\mathbb{R} \times \{2^{-2^k}\}$, and this tangent line does not pass through the origin and coincides locally at the interval $\tilde{I}_k$ with the graph of γ. Thus the claim follows from [25], since in any neighborhood of the origin the reciprocal of the distance between the curve and its tangent line is locally infinite. But we give a direct proof below.

Assume there exist a neighborhood U of the origin, $p < \infty$ and a constant $C_{p,U} \geq 0$ such that for every $f \in L^p$ the estimate

$$\|\mathcal{M}f\|_{L^p} \leq C_{p,U} \|f\|_{L^p}$$

holds true. Find a number $k \in \mathbb{N}$ such that $\tilde{I}_k \subset U$. Consider

$$f(x_1, x_2) = \frac{1}{|x_2|^{\frac{1}{2p}}} \mathbf{1}_{[-2^{2^k+4}, 2^{2^k+4}] \times [-1,1] \setminus \{0\}}(x_1, x_2).$$

Then obviously $\|f\|_{L^p} < \infty$ and $f \geq 0$. We show that for every $(z_1, z_2) \in [1,2] \times [1,2]$ the maximal function is identically ∞, and therefore $\mathcal{M}f$ is not in $L^p(\mathbb{R}^2)$.

Let $(z_1, z_2) \in [1,2] \times [1,2]$ and let $t_n = 2^{2^k}(z_2 + \frac{1}{n}) > 0$, $n \in \mathbb{N}$. We have

$$\begin{aligned}
\mathcal{M}f(z_1, z_2) &= \sup_{t>0} \left| \int\limits_U f(z - t(u, \gamma(u)))du \right| \\
&\geq \sup_{t>0} \int\limits_{\tilde{I}_k} f((z_1, z_2) - t(u, \gamma(u)))du \\
&\geq \int\limits_{\tilde{I}_k} f((z_1, z_2) - t_n(u, 2^{-2^k}))du.
\end{aligned}$$

For $u \in \tilde{I}_k$ we get

$$|z_1 - t_n u| \leq |z_1| + |t_n u| \leq 2 + t_n \leq 2 + 3 \cdot 2^{2^k} \leq 2^{2^k+4},$$

$$\left|z_2 - t_n 2^{-2^k}\right| = \left|z_2 - \left(z_2 + \frac{1}{n}\right)\right| = \frac{1}{n} < 1.$$

Therefore

$$\begin{aligned}
\mathcal{M}f(z_1, z_2) &\geq \int_{\tilde{I}_k} f((z_1, z_2) - t_n(u, 2^{-2^k}))du \\
&= \int_{\tilde{I}_k} |z_2 - t_n(2^{-2^k})|^{-\frac{1}{2p}} du \\
&= |\tilde{I}_k| n^{\frac{1}{2p}} \\
&= 2^{-k-9} n^{\frac{1}{2p}}.
\end{aligned}$$

The assertion follows when $n \longrightarrow \infty$.

6.2 Smooth finite type case

In this section we shall prove that Theorem 1.3.1 fails to be true if the function φ is not analytic. The following construction was suggested by my advisor Prof. Dr. D. Müller, and I wish to express my gratitude to him for this suggestion.
In fact, we can even disprove the L^p-regularity of the maximal average for $p > 2$ if the function is only assumed to be of finite type along every line.

Definition 6.2.1. Let $U \subseteq \mathbb{R}^n$ be an open set. Let $f \colon U \longrightarrow \mathbb{C}$ be a smooth function. The function f is said to be of finite type at $x_0 \in U$ along every line if for every $\eta \in S^{n-1}$ there exists $\varepsilon > 0$ such that the function

$$f_\eta \colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{C}, \quad t \longmapsto f(x_0 + t\eta)$$

is of finite type at 0.

Lemma 6.2.2. *For every $m \in \mathbb{N}$ there exists a smooth real-valued function ψ of finite type along every line at the origin such that the maximal operator*

$$\mathcal{M}f(z) = \sup_{t>0} \left| \int_U f(z - t(x_1, x_2, \psi(x_1, x_2)))dx \right|$$

is unbounded on $L^m(\mathbb{R}^3)$ for any neighborhood of the origin U.

Proof. Let $m \in \mathbb{N}$. Let γ be the curve defined in (6.2). Set

$$\psi(x_1, x_1) = \gamma(x_1) + (x_2 - x_1^2)^m.$$

We first check that ψ is of finite type along every line. Along the vertical line we see that $\psi(0,t) = t^m$, i.e. along the vertical line the function is of type m at the origin. For $\lambda \in \mathbb{R}$ consider $\psi(t, \lambda t) = \gamma(t) + (\lambda t - t^2)^m$. In this case the function is of type

$$2m\mathbf{1}_{\{0\}}(\lambda) + m\mathbf{1}_{\mathbb{R}\setminus\{0\}}(\lambda).$$

Assume there exists a neighborhood $U \subset \mathbb{R}^2$ of the origin such that $\mathcal{M}$ is bounded on $L^m(\mathbb{R}^3)$. After changing variables we get

$$\mathcal{M}f(\cdot) = \sup_{t>0} \left| \int_V f(\cdot - t(x_1, x_2 + x_1^2, \gamma(x_1) + x_2^m) dx \right|,$$

where V is some neighborhood of the origin.
There exists $\delta \in (0, \frac{1}{2})$ such that $(0,\delta) \times (0,\delta) \subseteq V$. There is a large $k \in \mathbb{N}$, $k \geq 3$, such that

$$[c(I_k) - 2^{-k-10}, c(I_k) + 2^{-k-10}] \times (0,\delta) = \tilde{I}_k \times (0,\delta) \subseteq (0,\delta)^2 \subseteq V.$$

Therefore for every $g \geq 0$ we get

$$\begin{aligned}\mathcal{M}g(\cdot) &\geq \sup_{t>0} \int_{\tilde{I}_k \times (0,\delta)} g(\cdot - t(x_1, x_2 + x_1^2, \gamma(x_1) + x_2^m))dx \\ &= \sup_{t>0} \int_{\tilde{I}_k \times (0,\delta)} g(\cdot - t(x_1, x_2 + x_1^2, c_k + x_2^m))dx,\end{aligned}$$

where $c_k = 2^{-2^k}$. For $N \geq 20$ let $A_N = [-\frac{3N}{c_k}, \frac{3N}{c_k}]^2 \times [-1,1]$. Set $g_N = \mathbf{1}_{A_N}$. Then obviously

$$\|g_N\|_{L^m(\mathbb{R}^3)}^m = 72 \cdot \frac{N^2}{(c_k)^2}.$$

Next, we show that for any $(z_1, z_2, z_3) \in [0, \frac{N}{4}]^2 \times [1, N]$ we get the estimate

$$\mathcal{M}g_N(z_1, z_2, z_3) \geq |\tilde{I}_k| c_k \delta z_3^{-\frac{1}{m}}.$$

Let $(z_1, z_2, z_3) \in [0, \frac{N}{4}]^2 \times [1, N]$ and set $\alpha = c_k \delta z_3^{-\frac{1}{m}}$. Observe that $\alpha \leq \frac{\delta}{2} < \delta$. For $t_0 = \frac{z_3}{c_k} > 0$ we get

$$\begin{aligned}\mathcal{M}g_N(z_1, z_2, z_3) &\geq \sup_{t>0} \int\limits_{\tilde{I}_k} \int\limits_0^{\delta} g_N\left((z_1, z_2, z_3) - t(x_1, x_2 + x_1^2, c_k + x_2^m)\right) dx_2 dx_1 \\ &\geq \int\limits_{\tilde{I}_k} \int\limits_0^{\alpha} g_N\left((z_1, z_2, z_3) - t_0(x_1, x_2 + x_1^2, c_k + x_2^m)\right) dx_2 dx_1.\end{aligned}$$

Observe that for every $(x_1, x_2) \in \tilde{I}_k \times (0, \alpha)$ we get

$$|z_1 - t_0 x_1| \leq |z_1| + |t_0 x_1| \leq \frac{N}{4} + \frac{z_3}{c_k} \cdot \frac{1}{2} \leq \frac{N}{4} + \frac{N}{2c_k} \leq \frac{3N}{c_k},$$

$$|z_2 - t_0(x_2 + x_1^2)| \leq \frac{N}{4} + t_0(1+1) = \frac{N}{4} + \frac{2N}{c_k} \leq \frac{3N}{c_k},$$

$$|z_3 - t_0(c_k + x_2^m)| = \frac{z_3}{c_k} x_2^m \leq \frac{z_3}{c_k} \alpha^m = (c_k)^{m-1} \delta^m \leq 1.$$

We conclude

$$\mathcal{M}g_N(z_1, z_2, z_3) \geq \int\limits_{\tilde{I}_k} \int\limits_0^{\alpha} 1 \; dx_2 dx_1 = |\tilde{I}_k| \alpha.$$

Thus

$$\begin{aligned}\|\mathcal{M}g_N\|_{L^m(\mathbb{R}^3)}^m &\geq \int\limits_0^{\frac{N}{4}} \int\limits_0^{\frac{N}{4}} \int\limits_1^{N} |\tilde{I}_k|^m (c_k)^m \delta^m \frac{1}{z_3} dz_3 dz_2 dz_1 \\ &= (c_k)^m \tfrac{N^2}{16} (2^{-k-9})^m \delta^m \log(N).\end{aligned}$$

Putting these estimates together we conclude

$$\frac{\|\mathcal{M}g_N\|_{L^m(\mathbb{R}^3)}^m}{\|g_N\|_{L^m(\mathbb{R}^3)}^m} \geq \frac{(c_k)^m \frac{N^2}{16} (2^{-k-9})^m \delta^m \log(N)}{72 \frac{N^2}{(c_k)^2}} = (c_k)^{m+2} \frac{(2^{-k-9})^m \delta^m}{1152} \log(N) \longrightarrow \infty$$

for $N \longrightarrow \infty$. $\square$

6.3 Remarks on the critical exponent $p = 2$

In this section we shall prove that the interval $(2, \infty]$ of the L^p-boundedness of the maximal average over a cylindrical surface in $\mathbb{R}^3$ cannot be improved, even for a very large class of dilations. For this purpose we consider the following situation.
Let $\emptyset \neq I \subseteq \mathbb{R}$ be any set in $\mathbb{R}$ and let $\psi_i \colon I \longrightarrow \mathbb{R}$, $i \in \{1, 2\}$, be continuous functions. We assume that there is a measurable set $A \subseteq \mathbb{R}$ with $|A| > 0$ and $A \subseteq \psi_1(I)$. Since for all $N \in \mathbb{N}$ the set $[-N, N] \cap A$ has a finite volume, we can assume that A is bounded and in particular $\infty > |A| > 0$. For $t \in I$ let

$$D_t^{\psi}(x_1, x_2, x_3) = (\psi_1(t)x_1, \psi_2(t)x_2, \psi_1(t)x_3).$$

Observe that clearly, the usual dilation $\psi_1(t) = \psi_2(t) = t$ on $(0, \infty)$ satisfies the above assumption.
For a neighborhood $V \subseteq \mathbb{R}^2$ of the origin and $m \geq 2$, consider the hypersurface

$$\Gamma^V = \{(x_1, x_2, x_1^m) : (x_1, x_2) \in V\}$$

and the average operator

$$Af(z,t) = \int_V f(z - D_t^{\psi}(x_1, x_2, x_1^m))dx, \quad z \in \mathbb{R}^3,\ t \in I,$$

initially defined on $\mathcal{S}$. Denote by $\mathcal{M}$ its associated maximal operator

$$\mathcal{M}f(z) = \sup_{t \in I} |Af(z,t)|, \quad f \in \mathcal{S}.$$

Lemma 6.3.1. *The maximal operator $\mathcal{M}$ is not bounded on $L^2(\mathbb{R}^3)$ for any neighborhood V of the origin.*

Proof. For any neighborhood V of the origin there exists $\delta > 0$ such that

$$(0, 2\delta) \times (0, \delta) \subseteq V.$$

For any positive function f the pointwise estimate

$$\begin{aligned}\mathcal{M}f(\cdot) &= \sup_{t\in I}\int\limits_{V} f(\cdot - D_t^{\psi}(x_1,x_2,x_1^m))dx \\ &\geq \sup_{t\in I}\int\limits_0^{2\delta}\int\limits_0^{\delta} f(\cdot - D_t^{\psi}(x_1,x_2,x_1^m))dx_2dx_1 \\ &= \delta^2\sup_{t\in I}\int\limits_0^{2}\int\limits_0^{1} f(\cdot - D_t^{\psi}(\delta x_1,\delta x_2,\delta^m x_1^m))dx_2dx_1\end{aligned}$$

holds true. In view of Lemma 2.2.1 and due to the simple observation that diagonal matrices commute, we conclude that it is sufficient to prove that the maximal operator

$$\tilde{\mathcal{M}}f(\cdot) = \sup_{t\in I}\left|\int\limits_0^{2}\int\limits_0^{1} f(\cdot - D_t^{\psi}(x_1,x_2,x_1^m))dx_2dx_1\right|$$

is unbounded on $L^2(\mathbb{R}^3)$. We write

$$\tilde{\mathcal{M}}f(\cdot) = \sup_{t\in I}\left|\int\limits_{-1}^{1}\int\limits_0^{1} f\big(\cdot - D_t^{\psi}(\Phi(x)+(1,0,1))\big)dx\right|,$$

where $\Phi(x) = \Phi(x_1,x_2) = (x_1+1, x_2, (x_1+1)^m) - (1,0,1)$. Let

$$T = \begin{pmatrix} \frac{1}{1-m} & 0 & \frac{1}{m-1} \\ 0 & 1 & 0 \\ \frac{m}{m-1} & 0 & \frac{1}{1-m}\end{pmatrix}.$$

Computation shows that $\det T = \frac{1}{1-m} \neq 0$. Simple computations also reveal that the commutation

$$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1\end{pmatrix}\cdot\begin{pmatrix}\mu_1 & 0 & \mu_2\\ 0 & \mu_3 & 0 \\ \mu_4 & 0 & \mu_5\end{pmatrix} = \begin{pmatrix}\mu_1 & 0 & \mu_2\\ 0 & \mu_3 & 0 \\ \mu_4 & 0 & \mu_5\end{pmatrix}\cdot\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1\end{pmatrix}$$

holds true. We also have

$$T\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{1-m} & 0 & \frac{1}{m-1}\\ 0 & 1 & 0\\ \frac{m}{m-1} & 0 & \frac{1}{1-m}\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Thus

$$\begin{aligned}D_t^{\psi}\cdot(\Phi(x)+(1,0,1)) &= T^{-1}\cdot T\cdot D_t^{\psi}\cdot(\Phi(x)+(1,0,1))\\ &= T^{-1}\cdot D_t^{\psi}\cdot(T\cdot\Phi(x)+(0,0,1)).\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}T\cdot\Phi(x) &= \begin{pmatrix}\frac{1}{1-m} & 0 & \frac{1}{m-1}\\ 0 & 1 & 0\\ \frac{m}{m-1} & 0 & \frac{1}{1-m}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\\ (x_1+1)^m-1\end{pmatrix}\\ &= \begin{pmatrix}\frac{x_1}{1-m}+\frac{((x_1+1)^m-1)}{m-1}\\ x_2\\ \frac{mx_1}{m-1}+\frac{((x_1+1)^m-1)}{1-m}\end{pmatrix}\\ &= \begin{pmatrix}x_1-x_1^2Q(x_1)\\ x_2\\ x_1^2Q(x_1)\end{pmatrix},\end{aligned}$$

where Q is a polynomial with $c:=|Q(0)|\neq 0$. Observe that for $\frac{1}{2}>\epsilon>0$ sufficiently small we have

$$\frac{c}{2}\leq|Q(x_1)|\leq 2c\quad\text{for any } x_1\in[0,\epsilon].$$

If we apply again Lemma 2.2.1 we see that it is sufficient to show that the maximal operator

$$\tilde{\tilde{\mathcal{M}}}f(\cdot)=\sup_{t\in I}\left|\int_{-1}^{1}\int_{0}^{1}f(\cdot-(\psi_1(t)(x_1-x_1^2Q(x_1)),\psi_2(t)x_2,\psi_1(t)(1+x_1^2Q(x_1))))dx_2dx_1\right|$$

is unbounded on $L^2(\mathbb{R}^3)$. Since A is bounded, there exists an integer N such that

$$A\subseteq[-N,N].$$

Let $\rho(s) = \exp(-s^2)$ and

$$g(z_1, z_2, z_3) = \frac{\rho(z_2)}{|z_3|^{\frac{1}{2}} \log \frac{1}{|z_3|}} \mathbf{1}_{[-3N,3N]\times\mathbb{R}\times[-\frac{1}{4},\frac{1}{4}]\setminus\{0\}}(z_1, z_2, z_3).$$

Then obviously $0 \le g \in L^2(\mathbb{R}^3)$ and

$$\tilde{\tilde{\mathcal{M}}} g(\cdot) \ge \sup_{t\in I} \int_0^\epsilon \int_0^1 g(\cdot - (\psi_1(t)(x_1 - x_1^2 Q(x_1)), \psi_2(t)x_2, \psi_1(t)(1 + x_1^2 Q(x_1))))dx_2 dx_1.$$

We assume ϵ to be small enough, namely $\epsilon \le \min\left\{\frac{1}{2}, \frac{1}{4}\sqrt{\frac{1}{4Nc}}\right\}$.
Let $(z_1, z_2, z_3) \in A \times A \times (A \setminus \{0\})$. Since $z_3 \in A \subseteq \psi_1(I)$, there exists $t_0 \in I$ such that $\psi_1(t_0) = z_3$.
Thus for each $(x_1, x_2) \in [0, \epsilon] \times [0, 1]$ the estimates

$$\begin{aligned}
|z_1 - \psi_1(t_0)(x_1 - x_1^2 Q(x_1))| &\le |z_1| + |\psi_1(t_0)x_1| + |\psi_1(t_0)|x_1^2|Q(x_1)| \\
&\le N + N(1 + \tfrac{2c}{4c}) \\
&\le 3N, \\
|z_3 - \psi_1(t_0)(1 + x_1^2 Q(x_1))| &= |z_3 x_1^2 Q(x_1)| \\
&\le \tfrac{N}{4} \cdot \tfrac{2c}{4Nc} \\
&\le \tfrac{1}{4}
\end{aligned}$$

hold true. Therefore

$$\begin{aligned}
\tilde{\tilde{\mathcal{M}}} g(z) &\ge \int_0^\epsilon \int_0^1 \frac{\rho(z_2 - \psi_2(t_0)x_2)}{|z_3 x_1^2 Q(x_1)|^{\frac{1}{2}} \log \frac{1}{|z_3 x_1^2 Q(x_1)|}} dx_2 dx_1 \\
&= C(z_2, \psi_2(t_0)) \int_0^\epsilon \frac{1}{|z_3 x_1^2 Q(x_1)|^{\frac{1}{2}} \log \frac{1}{|z_3 x_1^2 Q(x_1)|}} dx_1 \\
&= C(z_2, \psi_2(t_0)) \int_0^\epsilon \frac{1}{x_1 |z_3 Q(x_1)|^{\frac{1}{2}} \log \frac{1}{|z_3 x_1^2 Q(x_1)|}} dx_1.
\end{aligned}$$

Next, observe that the estimate

$$\begin{aligned}
\frac{1}{x_1|z_3Q(x_1)|^{\frac{1}{2}}\log\frac{1}{x_1^2|z_3Q(x_1)|}} &\geq \frac{1}{\sqrt{2c|z_3|}}\cdot\frac{1}{x_1\log\frac{1}{|z_3x_1^2Q(x_1)|}}\\
&\geq \frac{1}{\sqrt{2c|z_3|}}\cdot\frac{1}{x_1\log\frac{1}{x_1(\sqrt{|z_3||Q(x_1)|})^2}}\\
&= \frac{1}{2\sqrt{2c|z_3|}}\cdot\frac{1}{x_1\log\frac{1}{x_1\sqrt{|z_3||Q(x_1)|}}}\\
&\geq \frac{1}{2\sqrt{2c|z_3|}}\cdot\frac{1}{x_1\log\frac{1}{x_1\sqrt{|\frac{cz_3}{2}|}}}
\end{aligned}$$

holds true for every $x_1 \in (0,\epsilon]$. We obtain

$$\begin{aligned}
\tilde{\tilde{\mathcal{M}}}g(z) &\geq \frac{C(z_2,\psi_2(t_0))}{2\sqrt{2c|z_3|}}\int_0^{\epsilon}\frac{1}{x_1\log\frac{1}{x_1\sqrt{|\frac{cz_3}{2}|}}}dx_1\\
&= \frac{C(z_2,\psi_2(t_0))}{2\sqrt{c|z_3|}}\int_0^{\sqrt{\frac{c}{2}|z_3|}\epsilon}\frac{1}{x_1\log\frac{1}{x_1}}dx_1\\
&= \infty.
\end{aligned}$$

□

Appendix

Here we shall construct a smooth partition of unity adapted to certain homogeneous dilations.

Lemma A.0.2. *Let* $(\alpha_1, \ldots, \alpha_n) \in (0, \infty)^n$ *be a weight and let*

$$\delta_r(x_1, \ldots, x_n) = (r^{\alpha_1} x_1, \ldots, r^{\alpha_n} x_n), \quad r > 0,$$

be the associated one-parameter homogeneous dilation on $\mathbb{R}^n$*. Then there exists a smooth function* $\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ *with the properties*

$$0 \leq \rho \leq 1, \quad \operatorname{supp} \rho \subseteq \left\{ x \in \mathbb{R}^n : \ 1 \leq |x| \leq 4^{\max\limits_i \alpha_i} \cdot \sqrt{n} \right\},$$

such that for every $N \in \mathbb{N}$ *and for every* $x \in [-2^{-N\alpha_1}, 2^{-N\alpha_1}] \times \ldots \times [-2^{-N\alpha_n}, 2^{-N\alpha_n}] \setminus \{0\}$ *we have*

$$\sum_{j=N}^{\infty} \rho_j(x) = 1,$$

where $\rho_j(x) = \rho(\delta_{2^j}(x))$*.*

Proof. Let $\tilde{\rho}\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth radially decreasing function with the properties

$$0 \leq \tilde{\rho} \leq 1, \quad \tilde{\rho}_{\big|[-2^{\alpha_1}, 2^{\alpha_1}] \times \ldots \times [-2^{\alpha_n}, 2^{\alpha_n}]} \equiv 1, \quad \operatorname{supp} \tilde{\rho} \subseteq [-4^{\alpha_1}, 4^{\alpha_1}] \times \ldots \times [-4^{\alpha_n}, 4^{\alpha_n}].$$

Set

$$\rho(x) = \tilde{\rho}(x) - \tilde{\rho}(\delta_2(x)).$$

First, observe that $0 \leq \rho(x) \leq 1$ for every $x \in \mathbb{R}^n$. Next, notice that $|x| \leq 1$ implies $|x_i| \leq 1 \leq 2^{\alpha_i}$ for every $i = 1, \ldots, n$, and in particular, $|x_i| \leq 2^{\alpha_i}|x_i| \leq 2^{\alpha_i}$. This implies

$$\rho(x) = 1 - 1 = 0.$$

On the other hand, if $|x_i| > 4^{\alpha_i}$ for some $i = 1, \dots, n$, then

$$\tilde{\rho}(x) = 0 = \tilde{\rho}(\delta_2(x)).$$

Therefore $\rho(x) \neq 0$ implies $|x| \leq \sqrt{\sum_{i=1}^{n} 4^{2\alpha_i}} \leq 4^{\max_i \alpha_i} \cdot \sqrt{n}$. On the other hand, for every $M > N$ we have

$$\begin{aligned} \sum_{j=N}^{M} \rho_j(x) &= \sum_{j=N}^{M} \tilde{\rho}(\delta_{2^j}(x)) - \tilde{\rho}(\delta_{2^{j+1}}(x)) \\ &= \tilde{\rho}(\delta_{2^N}(x)) - \tilde{\rho}(\delta_{2^{M+1}}(x)). \end{aligned}$$

If $x \neq 0$, then $\lim\limits_{M\to\infty} |\delta_{2^{M+1}}(x)| = \infty$. It gives

$$\sum_{j=N}^{\infty} \rho_j(x) = \tilde{\rho}(\delta_{2^N}(x)) = \tilde{\rho}(2^{N\alpha_1}x_1, \dots, 2^{N\alpha_n}x_n).$$

Thus for every $x \in [-2^{-N\alpha_1}, 2^{-N\alpha_1}] \times \ldots \times [-2^{-N\alpha_n}, 2^{-N\alpha_n}] \setminus \{0\}$ the identity

$$\sum_{j=N}^{\infty} \rho_j(x) = 1$$

holds true. $\square$

Next, we shall prove the implications in (1.5). This proof was suggested by my advisor Prof. Dr. D. Müller, and I wish to express my gratitude to him for this suggestion.

Lemma A.0.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a neighborhood of the origin. Let $\phi\colon \Omega \to \mathbb{R}$ be a real-valued analytic function. Assume that*

$$\phi(0,0) = 0, \quad \nabla\phi(0,0) = (0,0).$$

Then for every $x^0 \in \Omega$

$$\operatorname{Hess}\phi(x^0) = 0_{\mathbb{R}^{2\times 2}} \Longrightarrow \nabla\phi(x^0) = (0,0) \Longrightarrow \phi(x^0) = 0$$

holds true, if the neighborhood Ω is sufficiently small.

Proof. First, observe that $\operatorname{Hess}\phi(x) = 0_{\mathbb{R}^{2\times 2}}$ is equivalent to $\Psi(x) = 0$, where

$$\Psi(x) = \left(\partial_1^2\phi(x)\right)^2 + \left(\partial_2^2\phi(x)\right)^2 + \left(\partial_1\partial_2\phi(x)\right)^2.$$

The function Ψ is analytic, since ϕ is analytic. It is a well known fact (see e.g. [10], [36]) that Ψ can be written in

$$\Psi(x_1, x_2) = U(x_1, x_2)x_1^A \prod_r (x_2 - r(x_1))^{n_r},$$

where U is an analytic function with $U(0,0) \neq 0$, A, $n_r \in \mathbb{N}_0$, and r are roots of Ψ. In a small neighborhood of the origin these roots can be expressed as Puiseux series, i.e. each r is a complex-valued analytic function in some fractional power $x_1^{\frac{1}{N}}$. More precisely, $r(x_1) = \tilde{r}(x_1^{\frac{1}{N}})$, where $\tilde{r}$ is a complex-valued analytic function with $\tilde{r}(0) = 0$. Of course, it is possible that the set of roots is empty.

Let $x^0 \in \Omega$. We can assume that x^0 lies in the right half-plane, i.e. $x_1^0 > 0$. Assume that $\operatorname{Hess}\phi(x^0) = 0_{\mathbb{R}^{2\times 2}}$. Therefore x^0 must lie on the curve $(t, r(t))$ for some root r. In particular, r is real-valued. Consider the curve

$$\gamma(t) = \phi(t, r(t)) - t\partial_1\phi(t, r(t)) - r(t)\partial_2\phi(t, r(t)), \quad t \in (0, \delta),$$

for some sufficiently small δ. Using the assumption that $\operatorname{Hess}\phi(t, r(t)) = 0_{\mathbb{R}^{2\times 2}}$, we easily conclude by computations that γ' vanishes identically. This implies that γ is constant on $(0, \delta)$. $\lim\limits_{t\to 0}\gamma(t) = 0$ implies that γ is identically zero on $(0, \delta)$. This is equivalent to

$$\phi(t, r(t)) = t\partial_1\phi(t, r(t)) + r(t)\partial_2\phi(t, r(t)).$$

Next, observe that the derivative of each curve $t \longmapsto \partial_j\phi(t, r(t))$, $j \in \{1, 2\}$, vanishes identically on some small interval $(0, \delta)$, hence $\operatorname{Hess}\phi(t, r(t)) = 0_{\mathbb{R}^{2\times 2}}$.

Using $\nabla\phi(0,0) = (0,0)$, we conclude that $\partial_j\phi(t, r(t)) \equiv 0$ for $t \in (0, \delta)$, which in turn implies that $\phi(t, r(t)) = 0$. $\square$

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit, abgesehen von der Beratung durch den Betreuer meiner Promotion, unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit oder Auszüge wurden bislang noch bei keiner anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegt oder veröffentlicht.

Kiel, den 21.05.2014

(Eugen Zimmermann)

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